XX. Quantum Restrictions on Cloning

Tutorial: *Quantum communication with photons,* Mario Krenn, https://arxiv.org/pdf/1701.00989.pdf

The Quantum Bit

In classical information and computation science, information is encoded in the most fundamental entity, the bit. Its two possible values 0 and 1 are physically realized in many ways, be it simply by mechanical means (as a switch), in solids by magnetic or ferroelectric domains (hard drives), or by light pulses (optical digital media). All of these methods have one thing in common—one state of the device mutually excludes the simultaneous presence of the other—the switch is **either on or off**.

The superposition principle entails one of the **most fundamental aspects of quantum physics**, namely to allow the description of a physical system as being in a **probabilistic combination of its alternative states**. This so-called Superposition of states not only provides all predictions for the outcome of physical measurement, it also has drastic consequences for the nature of the physical state that we ascribe to a system. Its most important direct implication is the so-called **no-cloning theorem**, which states that it is **impossible to obtain a perfect copy of a qubit in an unknown state without destroying the information content of the original**. The no-cloning theorem is the basis for the security of all quantum communication schemes described in the following sections.

To fully understand a qubit, it is important to distinguish between a coherent superposition and a mixture of possible states. For its use in quantum communication, it is important that a photon exists in a coherent superposition of its **possible states**. For example, a polarization qubit being in a coherent superposition of horizontal and vertical polarizations (with a certain phase relation) can be understood as a photon polarized diagonally at +45°. A polarizer set at this angle will **always transmit such a photon with 100% probability** (and zero probability when set to −45°). However, a photon in a **mixture (incoherent** superposition) of horizontal and vertical polarization states will be **transmitted with 50% probability**. **Quantum superpositions,** however, are **not limited to just two possible states.** The information carried by a photon is **potentially enormous**. While **polarization is necessarily a two-level (qubit) property,** other degrees of freedom of a photon such as its spatial or temporal structure can have many orthogonal levels. For example, a photon can exist in a coherent superposition of different paths coming out of a multi-port beam splitter. These types of superpositions are referred to as "**high-dimensional**" by virtue of their ability to **encode large amounts of information**.

Practical Introduction to Quantum Computing: From Qubits to Quantum Machine Learning, CERN

- The way to know the value of a qubit is to perform a measurement. However
- The result of the measurement is random
- When we measure, we only obtain one (classical) bit of information
- If we measure the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ we get 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$
- Moreover, the new state after the measurement will be $|0\rangle$ or $|1\rangle$ depending of the result we have obtained (wavefunction collapse)
- We cannot perform several independent measurements of $|\psi\rangle$ because we cannot copy the state (no-cloning theorem)

No Cloning Matrix Proof

Tutorial: *Physical_and_Theoretical_Chemistry,* Dr. Frank Rioux

Suppose a quantum copier exists which is able to carry out the following cloning operation.

$$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Next the cloning operation (using the same copier) is carried out on the **general qubit** shown below.

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) \\ \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) \\ \sin^2(\theta) \end{pmatrix}$$

Quantum transformations are unitary, meaning probability is preserved. This requires that the scalar products of the initial and final states must be the same.

$$\text{Initial state:} \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin(\theta)$$

$$\text{Final state:} \quad \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \sin^2(\theta)$$

It is clear from this analysis that quantum theory puts a significant restriction on copying.
Only states for which sin(θ) = 0 or 1 (0 and 90 degrees) can be copied by the original clone.

In conclusion, two quotes from Otters and Azure, *Physics Today*, February 2009, page 76.

Perfect copying can be achieved only when the two states are orthogonal, and even then one can copy those two states (...) only with a copier specifically built for that set of states.
In sum, one cannot make a perfect copy of an unknown quantum state, since, without prior knowledge, it is impossible to select the right copier for the job. That formulation is one common way of stating the no-cloning theorem.

An equivalent way to look at this (**See:** ***Quantum communication with photons***, Mario Krenn) , Page 8
Assume that a clone exists for the V-H polarization states.

$$\hat{C}|V\rangle|X\rangle = |V\rangle|V\rangle \qquad \hat{C}|H\rangle|X\rangle = |H\rangle|H\rangle \qquad \text{Equations 1 and 2.}$$

A diagonally polarized photon is a superposition of the V-H polarization states.

$$|D\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)$$

However, due to the linearity of quantum mechanics **the V-H clone cannot clone a diagonally polarized photon.**

$$\hat{C}|D\rangle|X\rangle = \hat{C}\frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)|X\rangle = \frac{1}{\sqrt{2}}\left(\hat{C}|V\rangle|X\rangle + \hat{C}|H\rangle|X\rangle\right) = \frac{1}{\sqrt{2}}\left(|V\rangle|V\rangle + |H\rangle|H\rangle\right)$$

$$\hat{C}|D\rangle|X\rangle \neq |D\rangle|D\rangle = \frac{1}{2}\left(|V\rangle|V\rangle + |V\rangle|H\rangle + |H\rangle|V\rangle + |H\rangle|H\rangle\right) \qquad \text{Equation 3}$$

The last line in equation (3) was obtained by using equations (1) and (2) for the cloning operator ˆC. The result is an entangled state that cannot be factorized into |DA> |DA>. If one were to measure either of the entangled photons individually, the result would be random, and certainly not |DA>. From this simple example it's clear that quantum cloning is not possible.