

IV. Dirac's Relativistic Matrix Mechanics

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In 1928, P.A.M. Dirac derived a **relativistic formulation of the quantum mechanics of fermions**. The equation is invariant under a Lorentz transformation and spin emerges as a natural consequence of the relativistic treatment.

The Relativistic Equation for the energy of a free particle has positive and negative roots, where the positive root signifies the energy of a particle and the negative root the energy of its antiparticle. This interpretation was confirmed experimentally with the discovery of the anti-electron (positron) in 1932 by Anderson.

$$E = \pm c \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2 c^2} \quad (1)$$

Dirac converted this to a soluble quantum mechanical operator by first writing the argument of the square root as a perfect square in order to get rid of the troubling radical operator which defied physical interpretation. In a second step he replaced energy and momentum with their differential operators, $E = -(\hbar/2\pi i)d/dt$ and $p_q = (\hbar/2\pi i)d/dq$, from non-relativistic quantum mechanics. **Math**

$$p_x^2 + p_y^2 + p_z^2 + m^2 \cdot c^2 = (\alpha_x \cdot p_x + \alpha_y \cdot p_y + \alpha_z \cdot p_z + \beta \cdot m \cdot c)^2$$

For this mathematical maneuver **to be valid** the following conditions must hold: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\alpha_x \cdot \alpha_y + \alpha_y \cdot \alpha_x = 0 \quad \alpha_x \cdot \alpha_z + \alpha_z \cdot \alpha_x = 0 \quad \alpha_x \cdot \beta + \beta \cdot \alpha_x = 0 \quad \alpha_y \cdot \alpha_z + \alpha_z \cdot \alpha_y = 0$$

$$\alpha_y \cdot \beta + \beta \cdot \alpha_y = 0 \quad \alpha_z \cdot \beta + \beta \cdot \alpha_z = 0 \quad p_x \cdot p_y = p_y \cdot p_x \quad p_x \cdot p_z = p_z \cdot p_x \quad p_y \cdot p_z = p_z \cdot p_y$$

In other words, the α s and β s **must anticommute** while the momentum operators as used above **must commute**. From the non-relativistic formulation of quantum mechanics it was already clear that the momentum operator pairs above did commute. In formulating a relativistic quantum mechanics, Dirac assumed the validity of the various multiplicative and differential operators of non-relativistic quantum mechanics for observable properties like energy, position and momentum.

Being aware of **Heisenberg's matrix approach** to **non-relativistic** quantum mechanics, Dirac realized the restrictions above regarding the α s and β could be satisfied by the following 4x4 matrices. These matrices are formed from the **Pauli X, Y, Z Basis Matrices** ($\sigma_x, \sigma_y, \sigma_z$) are **Unitary Hermitian Matrices**. Hermitian operators represent **observables** in quantum mechanics, so the Pauli matrices span the space of observables of the complex 2 dimensional Hilbert space. See Section IX. Stern-Gerlach Experiment for more on the Pauli Spin Matrices. I or Unit is the Identity operator.

Matrix Formulation

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{I} = \sum_i |i\rangle\langle i|$$

$n \times n$ matrix of a tensor product of a row vector and a column vector.

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1 := \sigma_x$$

$$\sigma_2 := \sigma_y$$

$$\sigma_3 := \sigma_z$$

$$\alpha_x := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_y := \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_z := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\beta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

First we show that $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I$

$$\alpha_x \cdot \alpha_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \alpha_y \cdot \alpha_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \alpha_z \cdot \alpha_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \beta \cdot \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we show that the **α s and β s anticommute: $\sigma_i \sigma_j + \sigma_j \sigma_i = 0$** **Noncommutability of measurements in QM.**

$$\alpha_x \cdot \alpha_y + \alpha_y \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_x \cdot \alpha_z + \alpha_z \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_x \cdot \beta + \beta \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_y \cdot \alpha_z + \alpha_z \cdot \alpha_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_y \cdot \beta + \beta \cdot \alpha_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_z \cdot \beta + \beta \cdot \alpha_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is now possible to write Dirac's relativistic energy equation as follows:

$$\sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E = \pm c (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc) \quad (2)$$

Before proceeding to the next step, the **substitution of the differential operators** for energy and momentum, it is instructive to look at the right side of the above equation which is a **4x4 Dirac relativistic energy operator**. Of course, the left side is a 4x4 matrix with **Energy, E, on the diagonal** and zeros everywhere else.

Note: The **arrow symbol** \rightarrow below is used in Math to evaluate an expression symbolically. Math returns the result as another expression in terms of the variable and symbols in the original problem.

$$-c \cdot (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta \cdot m \cdot c) \rightarrow \begin{bmatrix} -c^2 \cdot m & 0 & -c \cdot p_z & -c \cdot (p_x - p_y \cdot i) \\ 0 & -c^2 \cdot m & -c \cdot (p_x + p_y \cdot i) & c \cdot p_z \\ -c \cdot p_z & -c \cdot (p_x - p_y \cdot i) & c^2 \cdot m & 0 \\ -c \cdot (p_x + p_y \cdot i) & c \cdot p_z & 0 & c^2 \cdot m \end{bmatrix}$$

Substituting the traditional operators for energy and momentum yields,

$$-\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = - \left[\frac{c\hbar}{i} \left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z} \right) + \beta mc^2 \right] \Psi \quad (4)$$

Assuming the separability of the space and time coordinates [$\Psi(x,y,z,t) = \psi(x,y,z)\phi(t)$], **this four dimensional** differential equation is decoupled in to two differential equations. The time-dependent equation is easily solved and has the following solution.

$$\varphi(t) = e^{-i\frac{Et}{\hbar}} \quad (5)$$

The space part of the differential equation has the following form, with the relativistic Hamiltonian operating on the wave function.

$$-\left[\frac{c\hbar}{i}\left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}\right) + \beta mc^2\right]\psi = E\psi \quad (6)$$

As demonstrated above (eqn 3) the relativistic energy operator is a 4x4 matrix. Therefore, the wave function must be a four-component vector.

At this point Sherwin turns to the example of the free particle in the x-direction (see pages 292-295). He assumes that the solution has the form of a plane wave. However, as shown below substitution of the deBroglie equation in the plane wave equation yields the momentum eigenfunction in coordinate space.

$$\exp\left(i\frac{2\pi}{\lambda}x\right) \xrightarrow{\lambda = \hbar/p} \exp\left(i\frac{px}{\hbar}\right)$$

This means that this problem is extremely easy to solve in momentum space where the momentum operator is multiplicative. The calculation of the energy eigenvalues is straight forward using **Math Eigenvals** command. We simply ask for the eigenvalues of the relativistic energy operator as shown below.

$$\text{eigenvals}\left[-c \cdot (\alpha_x \cdot p_x + \beta \cdot m \cdot c)\right] \rightarrow \begin{pmatrix} c \cdot \sqrt{c^2 \cdot m^2 + p_x^2} \\ c \cdot \sqrt{c^2 \cdot m^2 + p_x^2} \\ -c \cdot \sqrt{c^2 \cdot m^2 + p_x^2} \\ -c \cdot \sqrt{c^2 \cdot m^2 + p_x^2} \end{pmatrix}$$

Calculation of the (unnormalized) eigenvectors is equally easy.

$$\text{eigenvecs}\left[-c \cdot (\alpha_x \cdot p_x + \beta \cdot m \cdot c)\right] = \begin{pmatrix} \frac{W + m \cdot c^2}{p_x \cdot c} & 0 & 0 & \frac{-W + m \cdot c^2}{p_x \cdot c} \\ 0 & \frac{W + m \cdot c^2}{p_x \cdot c} & \frac{-W + m \cdot c^2}{p_x \cdot c} & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad W = \sqrt{p_x^2 \cdot c^2 + m^2 \cdot c^4}$$