XXIII A. An Entanglement Swapping Protocol

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In the field of quantum information interference, superpositions and entangled states are essential resources. Entanglement, a non-factorable superposition, is routinely achieved when two photons are emitted from the same source, say a parametric down converter (PDC). Entanglement swapping involves the transfer of entanglement to two photons that were produced independently and never previously interacted. The **Bell states** are the **four maximally entangled two-qubit entangled basis for a four-dimensional Hilbert space** and play an essential role in quantum information theory and technology, including teleportation and entanglement swapping.

The Bell states are shown below.

$$\Phi_p = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Phi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \qquad \Phi_m = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Phi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\Psi_p = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Psi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \qquad \Psi_m = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right] \qquad \Psi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

A four-qubit state is prepared in which **photons 1 and 2 are entangled in Bell state Φ_p**, and **photons 3 and 4 are entangled in Bell state Ψ_m**. The state multiplication below is understood to be tensor vector multiplication.

$$\Psi = \Phi_p\cdot\Psi_m = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \qquad \Psi := \frac{1}{2}\cdot\begin{pmatrix}0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\end{pmatrix}^T$$

$$I := \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Four Bell state measurements are now made on photons 2 and 3 which entangles photons 1 and 4.

Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto Ψ_m.

$$\left(kronecker\left(I, kronecker\left(\Phi_p\cdot\Phi_p^T, I\right)\right)\cdot\Psi\right)^T = \begin{pmatrix}0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & -0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0\end{pmatrix}$$

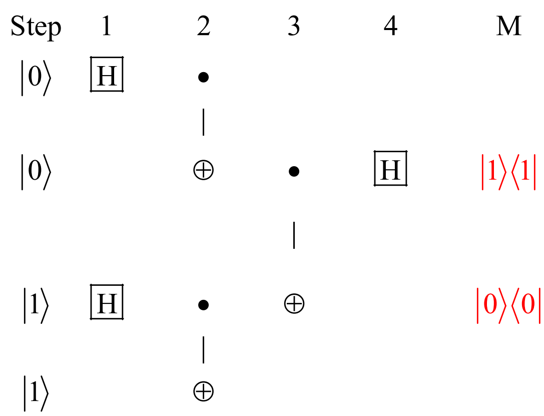
$$\left[\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]\right]^T = \frac{1}{4}\cdot\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix}$$

Projection of photons 2 and 3 onto Φ_m projects photons 1 and 4 onto Ψ_p.

$$\left(kronecker\left(I, kronecker\left(\Phi_m \cdot \Phi_m^T, I\right)\right) \cdot \Psi\right)^T = \begin{pmatrix} 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 \end{pmatrix}$$

$$\frac{1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^T = \frac{1}{4} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Here's a quantum circuit that accomplishes this entanglement swap.



$$\triangleright \quad \frac{1}{2\sqrt{2}}\left[|0101\rangle + |1100\rangle\right] = \frac{1}{2}|10\rangle_{23} \frac{1}{\sqrt{2}}\left[|01\rangle_{14} + |10\rangle_{14}\right]$$

Projection of photons 2 and 3 onto Ψ_p projects photons 1 and 4 onto $-\Phi_m$.

$$\left(kronecker\left(I, kronecker\left(\Psi_p \cdot \Psi_p^T, I\right)\right) \cdot \Psi\right)^T = \begin{pmatrix} 0 & 0 & -0.25 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^T = \frac{1}{4} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Finally, projection of photons 2 and 3 onto Ψ_m projects photons 1 and 4 onto Φ_p.

$$\left(kronecker\left(I, kronecker\left(\Psi_m \cdot \Psi_m^T, I\right)\right) \cdot \Psi\right)^T = \begin{pmatrix} 0 & 0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 & 0.25 & 0 & 0 \end{pmatrix}$$

$$\frac{-1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^T = \frac{1}{4} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$