

# VIA. Vector and Matrix Math

# Products of ket and bra-Spin vectors

Note: The **arrow symbol**  $\rightarrow$  below is used to indicate evaluating an expression symbolically. Symbolic returns the result as another expression in terms of the variable and symbols in the original problem.

Vector Inner Product:  $(a \ b) \cdot \begin{pmatrix} c \\ d \end{pmatrix} \rightarrow a \cdot c + b \cdot d$   $|\uparrow\rangle\langle\uparrow| \text{ UpUp} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1 \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Vector Outer Product:  $\begin{pmatrix} c \\ d \end{pmatrix} \cdot (ab) \rightarrow \begin{pmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{pmatrix}$   $|\uparrow\rangle\langle\downarrow| \text{ UpDn} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (0 \ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\text{tr} \left[ \begin{pmatrix} c \\ d \end{pmatrix} \cdot (a \ b) \right] \rightarrow a \cdot c + b \cdot d$   $|\downarrow\rangle\langle\uparrow| \text{ DnUp} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (1 \ 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|\downarrow\rangle\langle\downarrow| \text{ DnDn} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0 \ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Matrix-Vector Product:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a \cdot x + b \cdot y \\ c \cdot x + d \cdot y \end{pmatrix}$

$(x \ y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \rightarrow (a \cdot x + b \cdot y \quad c \cdot x + d \cdot y)$

Expectation Value:  $(x \ y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$

$(x \ y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow x \cdot (a \cdot x + b \cdot y) + y \cdot (c \cdot x + d \cdot y)$

$\text{tr} \left[ \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$

## Vector Tensor Product, $\otimes$ :

**A Tensor Product** is an operation between tensors. It creates a 3rd tensor.

**A Kronecker** is an operation between arrays.

$\text{tr} \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \cdot (x \ y) \right] \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$

## Matrix Product:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} a \cdot w + b \cdot y & a \cdot x + b \cdot z \\ c \cdot w + d \cdot y & c \cdot x + d \cdot z \end{pmatrix}$

## In Index notation

$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$

$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$

$VTP(V_1, V_2) := \begin{pmatrix} V_{1_0} \cdot V_{2_0} \\ V_{1_0} \cdot V_{2_1} \\ V_{1_1} \cdot V_{2_0} \\ V_{1_1} \cdot V_{2_1} \end{pmatrix}$

$VTP \left[ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right] \rightarrow \begin{pmatrix} a \cdot c \\ a \cdot d \\ b \cdot c \\ b \cdot d \end{pmatrix}$

**The tensor product of two column vectors, u and v,** is calculated by treating u as a column vector and v as a row vector and multiplying them. The result is a matrix.

**kroncker(M,N)** Multiplies matrix N by each element of matrix M, returning an M•N by M•N array.

**Arguments: M and N must be square matrices.**

**Pauli Spin Matrices**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} w & x \\ y & z \end{pmatrix} & b \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c \begin{pmatrix} w & x \\ y & z \end{pmatrix} & d \begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$MTP(A,B) := \begin{pmatrix} A_{0,0} \cdot B_{0,0} & A_{0,0} \cdot B_{0,1} & A_{0,1} \cdot B_{0,0} & A_{0,1} \cdot B_{0,1} \\ A_{0,0} \cdot B_{1,0} & A_{0,0} \cdot B_{1,1} & A_{0,1} \cdot B_{1,0} & A_{0,1} \cdot B_{1,1} \\ A_{1,0} \cdot B_{0,0} & A_{1,0} \cdot B_{0,1} & A_{1,1} \cdot B_{0,0} & A_{1,1} \cdot B_{0,1} \\ A_{1,0} \cdot B_{1,0} & A_{1,0} \cdot B_{1,1} & A_{1,1} \cdot B_{1,0} & A_{1,1} \cdot B_{1,1} \end{pmatrix}$$

$$TPP(X,Z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$MTP \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} w & x \\ y & z \end{pmatrix} \right] \rightarrow \begin{pmatrix} a \cdot w & a \cdot x & b \cdot w & b \cdot x \\ a \cdot y & a \cdot z & b \cdot y & b \cdot z \\ c \cdot w & c \cdot x & d \cdot w & d \cdot x \\ c \cdot y & c \cdot z & d \cdot y & d \cdot z \end{pmatrix} \quad S_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$MTP(X,Z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

**Matrix Eigenvalues and Eigenvectors (unnormalized):**

$$eigenvals \left( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right) \rightarrow \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

$$\left| \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} \right| = 0 \text{ solve, } \lambda \rightarrow \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}$$

$$kroncker(X,Z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$eigenvecs \left( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right) \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a-b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ solve, } y \rightarrow -x \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a+b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ solve, } y \rightarrow x \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

**TPP(A,B) Program for Tensor Product**

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TPP(A,B) :=
R ← rows(A) + rows(B)
C ← cols(A) + cols(B)
f(i,j) ← 0
T ← matrix(R,C,f)
for i ∈ 0..rows(A) - 1
  for j ∈ 0..cols(A) - 1
    for k ∈ 0..rows(B) - 1
      for l ∈ 0..cols(B) - 1
        Ti·rows(B)+k, j·cols(B)+l ← Ai,j·Bk,l
T
    
```

**Completeness Relations:**

$$\text{Black} \cdot \text{Black}^T + \text{White} \cdot \text{White}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Hard} \cdot \text{Hard}^T + \text{Soft} \cdot \text{Soft}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**TPP gives the same result as kroncker**

$$\text{AbsoluteSumArray}(kroncker(Z,X) - TPP(Z,X)) = 0$$