VIC. Tensor Products of Vector Spaces, Math ⊗

null vector, n $n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We use the matrix operation "kronecker ()" to calculate a Tensor Product, ⊗

The Math syntax for tensor multiplication of TWO 2-dimensional vectors. ⊗

$\psi(a,b) := submatrix(kronecker(augment(a,n),augment(b,n)),1,4,1,1)$

The Math syntax for tensor multiplication of THREE 2-dimensional vectors. ⊗

$\psi(a,b,c) := submatrix(kronecker(augment(a,n),kronecker(augment(b,n),augment(c,n))),1,8,1,1)$

kronecker(M, N)
Multiplies matrix N by each element of matrix M, returning an M•N by M•N array.
Arguments: M and N are square matrices.

The Tensor Product of Matrices (Refer back to Section VIA: Vector and Matrix Math) is also defined as:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mm} \end{pmatrix} \qquad A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$$

and **A ⊗ B** is therefore a matrix of **degree mn.**

Tutorial: *QUANTUM COMPUTING Gentle Introduction*, Eleanor Rieffel, Page 33

The *tensor product* $V \otimes W$ of two vector spaces V and W with bases $A = \{|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_n\rangle\}$ and $B = \{|\beta_1\rangle, |\beta_2\rangle, \dots, |\beta_m\rangle\}$ respectively is an nm-dimensional vector space with a basis consisting of the nm elements of the form $|\alpha_i\rangle \otimes |\beta_j\rangle$ where $\otimes$ is the tensor product, an abstract binary operator that satisfies the following relations:

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

$$(a|v\rangle) \otimes |w\rangle = |v\rangle \otimes (a|w\rangle) = a(|v\rangle \otimes |w\rangle).$$

Taking $k = \min(n, m)$, all elements of $V \otimes W$ have form

$$|v_1\rangle \otimes |w_1\rangle + |v_2\rangle \otimes |w_2\rangle + \cdots + |v_k\rangle \otimes |w_k\rangle$$

for some $v_i \in V$ and $w_i \in W$. Due to the relations defining the tensor product, such a representation is not unique. Furthermore, while all elements of $V \otimes W$ can be written

$$\alpha_1(|\alpha_1\rangle \otimes |\beta_1\rangle) + \alpha_2(|\alpha_2\rangle \otimes |\beta_1\rangle) + \cdots + \alpha_{nm}(|\alpha_n\rangle \otimes |\beta_m\rangle),$$

most elements of $V \otimes W$ cannot be written as $|v\rangle \otimes |w\rangle$, where $v \in V$ and $w \in W$. It is common to write $|v\rangle|w\rangle$ for $|v\rangle \otimes |w\rangle$.

QUANTUM COMPUTING EXPLAINED, David McMahon

Tensor Products - Chapter 4

In quantum mechanics we don't always work with single particles in isolation. In many cases, some of which are seen in the context of quantum information processing, it is necessary to work with multiparticle states. Mathematically, to understand multiparticle systems in quantum mechanics, it is necessary to be able to construct a **Hilbert space H** that is a composite of the independent Hilbert spaces that are associated with each individual particle. The machinery required to do this goes by the name of the **Kronecker or tensor product**. We consider the two-particle case. Suppose that H_1 and H_2 are two Hilbert spaces of dimension N_1 and N_2.

We can put these two Hilbert spaces together to construct a larger Hilbert space.
We denote this larger space by H and use **the tensor product operation** symbol ⊗. So we write $H = H_1 \otimes H_2$

In Math Notation: $H\_tensor\_I = kronecker(H, I)$

The dimension of the larger Hilbert space is the product of the dimensions of H_1 and H_2. Once again, we assume that $\dim(H_1) = N_1$ and $\dim(H_2) = N_2$. Then $\dim(H) = N_1 N_2$
Next we start getting down to business and learn how to represent state vectors in the composite Hilbert space.

REPRESENTING COMPOSITE STATES IN QUANTUM MECHANICS

A state vector belonging to H is the tensor product of state vectors belonging to H_1 and H_2.
We will show how to represent such vectors explicitly. For now we will just present some notation, sticking to the more abstract Dirac notation. Let $|\varphi\rangle \in H_1$ and $|\chi\rangle \in H_2$ be two vectors that belong to the Hilbert spaces used to construct H . We can construct a vector $|\psi\rangle \in H$ using the tensor product in the following way:

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$$

The tensor product of two vectors is linear. That is,

$$|\phi\rangle \otimes [|\chi_1\rangle + |\chi_2\rangle] = |\phi\rangle \otimes |\chi_1\rangle + |\phi\rangle \otimes |\chi_2\rangle$$
$$[|\phi_1\rangle + |\phi_2\rangle] \otimes |\chi\rangle = |\phi_1\rangle \otimes |\chi\rangle + |\phi_2\rangle \otimes |\chi\rangle$$

Moreover the tensor product is linear with respect to scalars

$$|\phi\rangle \otimes (\alpha|\chi\rangle) = \alpha|\phi\rangle \otimes |\chi\rangle$$

and vice versa. **To construct a basis** for the larger Hilbert space, we simply form the tensor products of basis vectors from the spaces H_1 and H_2. Let us denote the **basis of H_1 by $|u_i\rangle$** and the **basis of H_2 by $|v_i\rangle$**. Then it follows that we can construct a basis $|w_i\rangle$ for $H = H_1 \otimes H_2$ using $|w_i\rangle = |u_i\rangle \otimes |v_i\rangle$
Note that **the order of the tensor product is not relevant, meaning**

$$|\phi\rangle \otimes |\chi\rangle = |\chi\rangle \otimes |\phi\rangle$$

It is often cumbersome to write the ⊗ symbol. **Therefore you should be aware that**

the tensor product $|\varphi\rangle \otimes |\chi\rangle$ is often written more simply as **$|\varphi\rangle|\chi\rangle$, or even as $|\varphi\chi\rangle$.**

Quantum Bit and Quantum Register, *Fundamentals of Quantum Information*, Sagawa & Yoshida, Chap. 8

The qubit (quantum bit) can be implemented by any quantum system with **two states**.
The logical values 0 and 1 can be obtained, for example, by the direction of spin of an electron or an atom, the ground state and an excited state of an atom, and also the slit that a photon pass For example, the spin-up |↑> and the spin-down |↓> states of electrons can be assigned to | 0 > and | 1 > (see Bloch Sphere Section).

The set of qubits is called the quantum register, or simply the register. **An integer 6 is expressed as 110** in the binary system. In the quantum register, this can be represented as a direct product of **three states**,ψ
.

$$|6\rangle = |1\rangle \otimes |1\rangle \otimes |0\rangle$$

The state can be implemented by a three-bit register.
In general, a number of n-bits

$$a = 2^{n-1}a_{n-1} + 2^{n-2}a_{n-2} + \cdots + 2^0 a_0$$

can be expressed by a state of n-bit register,

$$|a\rangle = |a_{n-1}\rangle \otimes |a_{n-2}\rangle \otimes \cdots \otimes |a_1\rangle \otimes |a_0\rangle \equiv |a_{n-1}a_{n-2}\cdots a_1 a_0\rangle.$$

where ⊗ is the tensor product. See Section VI.

A quantum state of the n-bit register can be generalized to be a linear combination of states with numbers from a = 0 to a = $2^n - 1$:

Using the principles of quantum mechanics, **we can construct new types of reversible (unitary) gates which do not exist in the classical gates.** The Hadamard transformation H, for example, performs the operation,

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \qquad D_i^s(\alpha) = \cos\left(\frac{\alpha}{2}\right)\mathbf{1} - i\sin\left(\frac{\alpha}{2}\right)\sigma_i$$

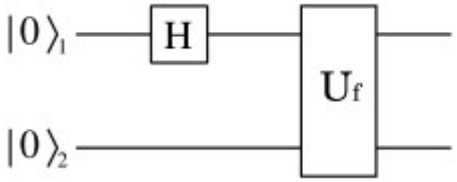
This gate is implemented by the product of the spin-rotation operator $D_y^s\left(\frac{\pi}{2}\right)$ and the spin operator σ_z.

$$H = D_y^s\left(\frac{\pi}{2}\right)\sigma_z = \left(\cos\left(\frac{\pi}{4}\right)\mathbf{1} - i\sin\left(\frac{\pi}{4}\right)\sigma_y\right)\sigma_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The rotation of spin can be achieved by the magnetic field such as the experimental set-up of Stern and Gerlach. Many quantum algorithms are based on so called quantum parallelism. As an defective algorithm by the quantum parallelism, we will show that the quantum computer can calculate the function f(x) for many different values of x by one instruction. As a simple example, let us take the function f(x), x ∈ {0,1}, f ∈ {0,1}, where both x and f(x) have 1 bit size. **We apply f(x) to the state |x,y> with two registers x and y**. The **first register x** is called **the data register** while the second register y is called the **target register.** The **operator U_f** acts on the state | x,y> as

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle$$

where y ⊗ f(x) means the **logical sum with mod 2.** The **operator U_f is called the Oracle operator** or "the black box."



$$|\psi\rangle = \frac{|0,f(0)\rangle + |1,f(1)\rangle}{\sqrt{2}}$$

We apply first the Hadamard transformation to the state | 0> of the data register.
Secondly we apply U_f to obtain the state

$$|\psi\rangle = U_f H|0,0\rangle = U_f \frac{1}{\sqrt{2}}\{|0,0\rangle + |1,0\rangle\} = \frac{1}{\sqrt{2}}\{|0,f(0)\rangle + |1,f(1)\rangle\}$$

The state arises as a result by the algorithm based on the **quantum parallelism**; the state $|\psi>$ **contains two results f(0) and f(1) as a linear combination**. This is the fundamental difference from the parallelism in the classical computer. In the parallelism in the classical circuits, **the different values of f(x)** are **calculated in separate circuits**. In the quantum parallelism, on the other hand, **all the values of f(x)** are calculated in a **single circuit**. The concept of the quantum parallelism holds not only with a single bit but also with n data bits. Let us write $H^{\otimes n}$ to express the Hadamard transformation acting on each bit of the n bit state $|0>^{\otimes n} = |00\cdots0>$. Note: $^{\otimes n}$ is an exponent.

Then we have a linear combination of 2^n states: where $\otimes$ is the tensor product.

$$
\begin{aligned}
|\psi\rangle = H^{\otimes n}|0\rangle &= H \otimes H \otimes \cdots \otimes H|00\cdots0\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{\sqrt{2^n}}(|00\cdots0\rangle + |00\cdots1\rangle + \cdots + |11\cdots1\rangle) \\
&= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle. \qquad (8.12)
\end{aligned}
$$

The above result shows that a **single operation of U_f** produces a state which **contains all of the 2^n values of the function f(x).** This conclusion is based on nothing but the principles of quantum mechanics. However when we consider the problem of quantum mechanical measurement, it is not very clear how effective the quantum parallelism is compared with the classical parallelism. In reality, the output register f(x) is determined by the observation of value x in the input register. This procedure implies that one output is determined by one input data just like the classical computer.

Two Different Formulations of Quantum Mechanics

Schrödinger's Wave Mechanics Formulation

Schrödinger's equation for the one-dimensional harmonic oscillator will be set up in **both coordinate and momentum space** using the information in the table. Schrödinger's equation is the quantum mechanical energy eigenvalue equation, and for the harmonic oscillator it looks like this initially, the information in the table.

$$\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}\right]|\Psi\rangle = E|\Psi\rangle$$

Energy observables: Hamiltonian in Sch. Eq. gives the time evolution of any system. Sch. Equation is the energy eigenvalue equation.

$$\hat{H}|E\rangle = E\,|E\rangle$$

The term in brackets on the left is the classical energy written as an operator without a commitment to a representation (position or momentum) for the calculation. See Sections **XXI and XXII** for solutions to the Schrödinger's equation. It is now necessary to explore the meaning of $<x|\Psi>$. It is the probability amplitude that a system in the state $|\Psi>$.

There are, of course, many formulations of quantum mechanics, and all of them develop quantum mechanical principles in different ways from diverse starting points, but they are all formally equivalent. In the present approach the key concepts are de Broglie's hypothesis, and the eigenvalue equations expressed in the momentum and coordinate representations, respectively.

Another formulation: Heisenberg or Matrix Mechanics

It identifies the commutation relation of equation as the basis of quantum theory, and adopts operators for position and momentum that satisfy the equation:

$$[\hat{p},\hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} = \frac{\hbar}{i}$$

Quantum mechanics based "exclusively on relationships between quantities that in principle are observable."

This Equation can be confirmed in both coordinate and momentum space for any state function |Ψ> **using the matrix operators** .

$$\langle x|(\hat{p}\hat{x} - \hat{x}\hat{p})|\Psi\rangle = \frac{\hbar}{i}\left(\frac{d}{dx}x - x\frac{d}{dx}\right)\langle x|\Psi\rangle = \frac{\hbar}{i}\langle x|\Psi\rangle$$

$$\langle p|(\hat{p}\hat{x} - \hat{x}\hat{p})|\Psi\rangle = i\hbar\left(p\frac{d}{dd} - \frac{d}{dp}p\right)\langle p|\Psi\rangle = \frac{\hbar}{i}\langle p|\Psi\rangle$$

The meaning associated with these equations is that the observables associated with **non-commuting operators cannot simultaneously have well-defined values**. This, of course, is just another statement of the **uncertainty principle.**

The famous double-slit experiment illustrates the uncertainty principle in a striking way. To illustrate this it is mathematically expedient to begin with infinitesimally thin slits. Later this restriction will be relaxed. A screen with **infinitesimally thin slits at x_1 and x_2** projects the incident beam into a linear superposition of **position** eigenstates

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|x_1\rangle + |x_2\rangle]$$

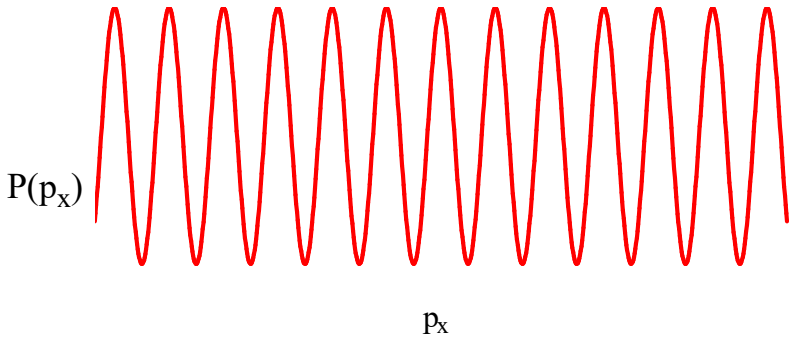
Expressing this state in the coordinate representation yields the following superposition of Dirac delta functions, δ.

$$\langle x|\Psi\rangle = \frac{1}{\sqrt{2}}[\langle x|x_1\rangle + \langle x|x_2\rangle] = \frac{1}{\sqrt{2}}[\delta(x - x_1) + \delta(x - x_2)]$$

According to the uncertainty principle **this localization** of the incident beam in coordinate space is accompanied by a **delocalization** of the x-component of the momentum, p_x. This can be seen by projecting |Ψ> onto momentum space

$$\langle p_x|\Psi\rangle = \frac{1}{\sqrt{2}}[\langle p_x|x_1\rangle + \langle p_x|x_2\rangle] = P(p_x) := \frac{1}{2\sqrt{\pi\cdot h}}\cdot\left(exp\left(-i\cdot p_x\cdot x_1\right) + exp\left(-i\cdot p_x\cdot x_2\right)\right)$$

The momentum probability distribution in the x-direction, $P(p_x) = |\langle p_x|\Psi\rangle|^2$ reveals the required spread in momentum, plus the interesting **interference pattern in the momentum distribution** that will ultimately be projected onto the detection screen. **The detection screen** is actually measuring the **x-component of the momentum**.



All nano (atom) sized objects "exist" fundamentally as a Wave-Particle Duality Amplitude