VIIIA. Bell States: Two-Electron Entangled State (Quantum Entanglement)

Imagine two electrons in a singlet spin state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$$

This is an entangled state where you **cannot assign a definite spin to either electron**.
Even though there are "two" electrons, they are **part of a single quantum system.**
If you measured one spin to be up, the other is instantly known to be down (nonlocally).

In quantum theory: Two electrons form a unique state (with entanglement and Pauli exclusion rules).

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$$

this was reintroduced into the mainstream literature of quantum entanglement with the following alternative formulations.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle) \quad |\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$$

$|\psi\rangle = |+1\rangle|-1\rangle + |-1\rangle|+1\rangle$

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$$

this is a superposition of two tensor products

$$n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Step-by-Step: Singlet State as Tensor Product (TP)

each electron has two spin states:

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (spin-up)}$$

$$|\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ (spin-down)}$$

Tensor Product for 1D Row Vectors

$$TP(a,b) := submatrix(kronecker(augment(a,n), augment(b,n)), 0, 3, 0, 0)$$

$$Up := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Dn := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Full Matrix Form (Tensor Product Expansion)

We can expand it using the Kronecker product:

$$|\uparrow\rangle \otimes |\downarrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$TP(Up, Dn) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad TP(Dn, Up) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|\downarrow\rangle \otimes |\uparrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

So the full singlet state becomes:

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\psi_{singlet}(\alpha, \beta) := \frac{1}{\sqrt{2}} \cdot (TP(\alpha, \beta) - TP(\beta, \alpha))$$

$$\psi_{singlet}(Up, Dn) = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

Basis Labels

This corresponds to the spin basis:

$$|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix} \text{ (spin-up)} \qquad |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix} \text{ (spin-down)}$$

$$|\uparrow\uparrow\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |\uparrow\downarrow\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad |\downarrow\uparrow\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad |\downarrow\downarrow\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Singlet state:

A singlet in quantum mechanics—especially in the context of spin—is a special kind of entangled state of **two particle**s (like electrons) **where the total spin angular momentum is zero.**

For two spin-½ particles, like electrons, the singlet state is:

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

The singlet state is antisymmetric under particle exchange,
which aligns with the Pauli exclusion principle for fermions.
It has total spin angular momentum
S = 0, meaning it is rotationally invariant.

Singlet Defined

For two spin-½ particles, like electrons, the singlet state is:

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$$

$$TP(Up, Dn) - TP(Dn, Up) = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

This is a **superposition** of **one particle being spin-up and the other being spin-down**, but in **an antisymmetric combination**.

Total Spin is Zero

1. The total spin angular momentum

S_{total} satisfies: $S^2 |\Psi\text{singlet}\rangle = 0$

2. ***Antisymmetric Under Particle Exchange***

Swapping the two particles flips the sign:

$$P_{12}|\Psi_{\text{singlet}}\rangle = -|\Psi_{\text{singlet}}\rangle$$

This makes it suitable for fermions like electrons, which obey the Pauli exclusion principle.

Maximally Entangled

You cannot describe one particle's spin without referencing the other.
Measurement of one spin instantly determines the other—no matter the distance between them.

Rotationally Invariant

The singlet looks the same in any coordinate system. That means the state remains unchanged (up to a global phase) under any rotation, R:

$$R|\Psi_{\text{singlet}}\rangle = |\Psi_{\text{singlet}}\rangle$$

EPR-Type Measurement Setup

you have two entangled electrons in the singlet state:

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle \otimes |\downarrow_z\rangle - |\downarrow_z\rangle \otimes |\uparrow_z\rangle\right)$$

You send one electron to Alice, and the other to Bob. Each measures spin along a chosen axis (e.g., z, x, or some angle θ).

Predicted Meassurement Correlations

When Alice and Bob measure spin along the same direction, say both along the z-axis:
If Alice measures ↑, Bob will definitely measure ↓.

If Alice measures ↓, Bob will definitely measure ↑.
This means:

P(same spin)=0, P(opposite spin)=1

So the spin measurements are perfectly anti-correlated.

Bell Inequality Violation

In classical (local hidden variable) theories, Bell derived an inequality that limits the strength of correlations.
Quantum mechanics violates this inequality using the singlet state. For specific angles, like:

- $\vec{a} = 0^\circ$,
- $\vec{a}' = 45^\circ$,
- $\vec{b} = 22.5^\circ$,
- $\vec{b}' = -22.5^\circ$,

The quantum predictions give values that violate Bell's inequality, proving:

- The correlations can't be explained by any local hidden variables.
- The singlet state contains nonlocal entanglement.

Feature	Singlet State Prediction
Same-axis measurement	Perfect anti-correlation
Arbitrary angles	Correlation = −cos(θ)
Bell test	Violates local realism

Entanglement is locally created.

Like tearing a photograph in two and sending it to two different locations. They are nonlocally correlated, but correlation was locally created. But it is a correlation that is stronger than for non-quantum particles. But it is just a property of how quantum states combine.