Entanglement

Tutorial: *Overview On quantum entanglement*, J. Ladvánszky, Ericsson Hungary

Quantum entanglement is a physical phenomenon which occurs when pairs or groups of particles are generated, interact, or share physical proximity in ways such that the **quantum state of each particle cannot be described independently of the state of the other(s)**, even when the particles are separated by a large distance—instead, a quantum state must be described for the system as a whole. Measurements of physical properties such as position, momentum, spin, and polarization, performed on entangled particles are found to be correlated. For example, if a pair of particles is generated in such a way that their total spin is known to be zero, and one particle is found to have clockwise spin on a certain axis, the spin of the other particle, measured on the same axis, will be found to be counterclockwise, as is to be expected due to their entanglement.

An entangled system is defined to be **one whose quantum state cannot be factored as a product of states of its local constituents**; that is to say, they are not individual particles but are an inseparable whole. In entanglement, **one constituent cannot be fully described without considering the other(s)**. Note that the state of a composite system is always expressible as a sum, or superposition, of products of states of local constituents; it is entangled if this sum necessarily has more than one term.

Just as classical bits are the fundamental building block of classical computers, quantum bits--or "qubits"--are the basic unit of information in quantum computers. Whereas classical bits can either have a value of 0 or 1, qubits can be in a combination of the states |0> and |1>. If the qubit is **not exactly in the state |0> or in the state |1>,** but rather in **some combination of both** the |0> and |1> states, then we say the qubit is in a "superposition" of the two states. The paradox is that a measurement made on either of the particles apparently collapses the state of the entire entangled system—and does so instantaneously.

Quantum mechanical framework

PureStates: Consider **two noninteracting systems A and B,** with respective Hilbert spaces H_A and H_B. The Hilbert space of the composite system is the tensor product $\mathbf{H_A \otimes H_B}$

If the first system is in state $|\psi\rangle_A$ and the second in state $|\phi\rangle_B$, the state of the composite system is $|\psi\rangle_A \otimes |\phi\rangle_B$

States of the composite system that can be represented in this form are called *separable* states, or *product* states. Not all states are separable states (and thus product states). Fix a basis $|i\rangle_A$ for H_A and a basis $|j\rangle_B$ for H_B. The most general state in $H_A \otimes H_B$ is of the form $$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

This state is separable if there exist vectors $[c_i^A],[c_j^B]$ so that $c_{ij}=c_i^A c_j^B$, yielding $|\psi\rangle_A=\sum_i c_i^A|i\rangle_A$ and $|\phi\rangle_B=\sum_j c_j^B|j\rangle_B$.

It is inseparable if for any vectors $[c_i^A],[c_j^B]$ at least for one pair of coordinates c_i^A, c_j^B we have $c_{ij}\neq c_i^A c_j^B$.

If a state is inseparable, it is called an entangled state.

For example, given two basis vectors $\{|0\rangle_A,|1\rangle_A\}$ of H_A and two basis vectors $\{|0\rangle_B,|1\rangle_B\}$ of H_B, the following is an entangled state: $\frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right)$

If the composite system is in this state, it is **impossible** to attribute to either system A or system B **a definite pure state**. Another way to say this is that while the von Neumann entropy of the whole state is zero (as it is for any pure state), the entropy of the subsystems is greater than zero. In this sense, the systems are "entangled". This has specific empirical ramifications for interferometry. It is worthwhile to note that the above example is one of four Bell states, which are (maximally) entangled pure states (pure states of the $H_A \otimes H_B$ space, but which cannot be separated into pure states of each H_A and H_B).

Measurement:
When we measure a qubit that is **exactly in the |0> state or |1> state**, then the qubit **will remain in that state.** However, if we measure a qubit that is in a **superposition**, then this **superposition collapses** from being in a combination of two states to being exactly in one of the two states. We cannot predict if the superposition will collapse into either the |0> or the |1> state with certainty, but we **can only know the probabilities** of being measured in either of the two states.

Entanglement:
It turns out that quantum states can extend over multiple qubits. When two or more qubits are entangled, measuring one of the qubits has an effect on the probability distributions of collapse for the other qubits. A Bell pair is a special 2-qubit quantum state, with properties that make it especially useful for certain applications.

Two indistinguishable particles with spin 1/2

LibreTexts Physics, Graeme Ackland, U of Edinburgh

If we have two identical fermions of spin 1/2, confined in the same region, what is the appropriate wavefunction? In the scattering case we could measure spins far from the interaction, and if we knew that the total spins is conserved, spins can be associated with each particle. In the bound state we cannot tell which particle we are measuring, so the ket must contain both spin and spatial wavefunctions of both particles.

Assuming the spins do not interact, we can separate the two-particle spin wavefunction into $\sigma(1,2)=\sigma_1\sigma_2$. We also know the appropriate one particle basis states $\uparrow 1, \downarrow 1, \uparrow 2, \downarrow 2$, where $\uparrow 1$ represents "particle 1" in spinor state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The combinations for indistinguishable particles are then:

$$\uparrow_1\uparrow_2, \quad \downarrow_1\downarrow_2, \quad (\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)/\sqrt{2}, \quad (\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)/\sqrt{2}$$

Operating on these with $\hat{P}_{12}$ yields eigenvalues 1, 1, 1 and -1 respectively. $S^2=S(S+1)$ yields 2, 2, 2 and 0, S_z yields 1, -1, 0 and 0. Thus the demands of indistinguishability couples the spins of two identical particles into a triplet (S=1) and a singlet (S=0). The spin-1 vector has three possible M_s component values - hence the triplet.

What are the Bell states?
The term Bell pairs actually describes **one of four entangled two qubit quantum states**, known collectively as the four "Bell states." Two of the Bell states give an **equal superposition** such that both of the qubits end up in the same state when measured, with a **50% chance that both will be in either the |0> or |1> state**. The other two Bell pairs give an equal superposition such that both of the qubits end in **opposite states when measured**. This means that if the first qubit is measured in |0>, then the second qubit will be measured in |1> and vice versa.

Definition of Entanglement (Bell States)

Bell Pair Symbol	Mathematical Representation	
$\|\Phi^+\rangle$	$= \frac{1}{\sqrt{2}}(\|0\rangle_A \otimes \|0\rangle_B + \|1\rangle_A \otimes \|1\rangle_B)$	Same States
$\|\Phi^-\rangle$	$= \frac{1}{\sqrt{2}}(\|0\rangle_A \otimes \|0\rangle_B - \|1\rangle_A \otimes \|1\rangle_B)$	
$\|\Psi^+\rangle$	$= \frac{1}{\sqrt{2}}(\|0\rangle_A \otimes \|1\rangle_B + \|1\rangle_A \otimes \|0\rangle_B)$	Opposite States
$\|\Psi^-\rangle$	$= \frac{1}{\sqrt{2}}(\|0\rangle_A \otimes \|1\rangle_B - \|1\rangle_A \otimes \|0\rangle_B)$	

In each of the Bell pairs, if one of the two qubits is measured, then we know exactly what the other qubit will be when it is measured. Let's consider one of |Φ+> or |Φ->. These are the two Bell pairs where both of the qubits must end up in the **same state** when measured. If we measure the first qubit in the state |0>, we know that the second qubit must be in |0> when we measure it. Even if these two qubits were on opposite ends of the globe and we measured them immediately one after another, if we found |0> |0> for the first qubit, then the second qubit would have to also be in the |0> |0> state. **Entanglement is necessary for a wide range of quantum networking protocols**, and Bell pairs are the most widely used entangled states for these protocols.

<u>The Bell states are maximally entangled superpositions of two-particle states</u>. Consider two spin-1/2 particles created in the same event. **There are four maximally entangled wave functions** representing their collective spin states. Each particle has two possible spin orientations and therefore the composite state is represented by a 4-vector in a

$$|\Phi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle + |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \qquad \Phi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$|\Phi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle - |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \qquad \Phi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$|\Psi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \qquad \Phi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$|\Psi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \qquad \Phi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$|\Phi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle + |\downarrow_1\rangle|\downarrow_2\rangle]$$

$$\Phi_p := \frac{1}{\sqrt{2}}\cdot(TP(Up, Up) + TP(Dn, Dn)) \qquad \Phi_p = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}$$

$$|\Phi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle - |\downarrow_1\rangle|\downarrow_2\rangle]$$

$$\Phi_m := \frac{1}{\sqrt{2}}\cdot(TP(Up, Up) - TP(Dn, Dn)) \qquad \Phi_m = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}$$

$$|\Psi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle]$$

$$\Psi_p := \frac{1}{\sqrt{2}}\cdot(TP(Up, Dn) + TP(Dn, Up)) \qquad \Psi_p = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}$$

$$|\Psi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle$$

$$\Psi_m := \frac{1}{\sqrt{2}}\cdot(TP(Up, Dn) - TP(Dn, Up)) \qquad \Psi_m = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}$$

Bell States Continued

The wave functions are not separable
and consequently the entangled particles represented by these wave functions
do not have separate identities or individual properties,
they behave like a **single entity**.

Individually the spins don't have a definite polarization, yet there is a definite spin orientation relationship between them. For example, if the spin orientation of particle 1 is learned through measurement, the spin orientation of particle 2 is also immediately known ***no matter how far away it may be***. ("*Spooky Action at a Distance"*)
Entanglement implies nonlocal phenomena which in the words of Nick Herbert are "unmediated, unmitigated and immediate."

Spooky-Action-at-a-Distance
(Spukhafte Fernwirkung)

The Bell states can be generated from two classical bits
with the use of a quantum circuit involving a **Hadamard (H) gate, the identity (I) and a controlled-not gate (CNOT)** as shown in Section IV and below .
The H gate operates on the top bit creating a superposition which controls the operation of the CNOT gate.
The classical state on the left also serves as an index for the Bell state created from it: 0, 1, 2, 3.

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Function: kronecker(M, N): Multiplies matrix N by each element of matrix M, returning an M•N by M•N array. Arguments: M and N **are square matrices.**

The controlled-NOT gate, CNOT, acts on a pair of qubits, with one acting as 'control' and the other as 'target'. It performs a **NOT** on the target, **if and only if, the control bit is |1**>. If the control qubit is in a superposition, this gate creates entanglement.

Bell State Generator, BSG

$$BSG := CNOT \cdot kronecker(H, I)$$

$$OO := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad O1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad '10 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad '11 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$BSG \cdot OO = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \qquad BSG \cdot '10 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \qquad BSG \cdot O1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \qquad BSG \cdot '11 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

It is not surprising that a Bell state measurement (BSM) is just the reverse of this process.

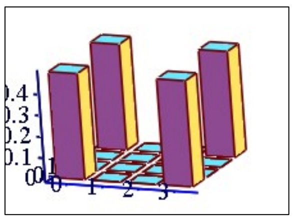
$$BSM := BSG^T$$

$$BSM \cdot BSG \cdot OO = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad BSM \cdot BSG \cdot O1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad BSM \cdot BSG \cdot '10 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad BSM \cdot BSG \cdot '11 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Individually the particles appear to behave like classical mixtures, but collectively they exhibit quantum correlations. We will now look at this issue from another perspective.

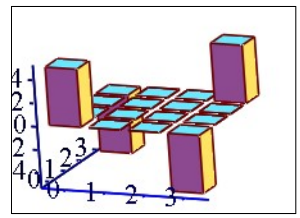
The density operators (|Ψ><Ψ|) are calculated for each of the Bell states

$$\Phi_p \cdot \Phi_p^{\,T} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$



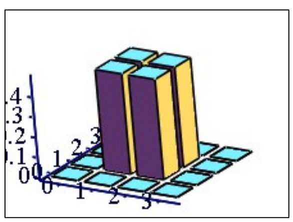
$\Phi_p \cdot \Phi_p^{\,T}$

$$\Phi_m \cdot \Phi_m^{\,T} = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$



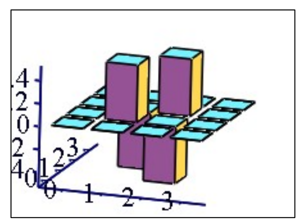
$\Phi_m \cdot \Phi_m^{\,T}$

$$\Psi_p \cdot \Psi_p^{\,T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$\Psi_p \cdot \Psi_p^{\,T}$

$$\Psi_m \cdot \Psi_m^{\,T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$\Psi_m \cdot \Psi_m^{\,T}$

We also demonstrate that the Bell states are pure states by showing that: **(|Ψ><Ψ|)^2 = |Ψ><Ψ|**

The Bell states are pure (see above) because for them **the trace of the square of the density operator equals 1.**

$$\left(\Phi_p \cdot \Phi_p^{\,T}\right)^2 = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix} \qquad \left(\Phi_m \cdot \Phi_m^{\,T}\right)^2 = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix} \qquad tr\left(\Phi_m \cdot \Phi_m^{\,T}\right)^2 = 1$$

$$tr\left(\Phi_p \cdot \Phi_p^{\,T}\right)^2 = 1$$

$$\left(\Psi_p \cdot \Psi_p^{\,T}\right)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \left(\Psi_m \cdot \Psi_m^{\,T}\right)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$tr\left(\Psi_p \cdot \Psi_p^{\,T}\right)^2 = 1 \qquad tr\left(\Psi_m \cdot \Psi_m^{\,T}\right)^2 = 1$$