VIIIB. The Bloch Sphere: Is a physical representation of all possible qubit states.

Each qubit is in its essence a vector on Bloch's sphere.

The Three ZXY Axes: State: |0> Z (Up), State: (|+>) X Front (+), State: (|i>) Y Side (i)

0 State (Z Up)	1 State (Z Down)	+ State (X Front)	- State (X Back)	i State (Y Right)	-i State (Y Left)
$Z_u := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$Z_d := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$X_f := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$X_b := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$Y_r := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$	$Y_l := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Create a Bloch Sphere (XYZ) With Angles: θ, ϕ to Represent Any Qubit State

NOTE: Index i is an i_{dot}

X,Y,Z are Xdot dot, Ydot dot' Z dot dot

$$n := 50 \qquad i := 0..n \qquad j := 0..n \qquad \theta_i := \pi \cdot \frac{i}{n} \qquad \phi_j := 2\pi \cdot \frac{j}{n}$$

$$\theta 1 := \frac{\pi}{2} \qquad \phi 1 := 0 \qquad X_{i,j} := \sin(\theta_i) \cdot \cos(\phi_j) \qquad Y_{i,j} := \sin(\theta_i) \cdot \sin(\phi_j) \qquad Z_{i,j} := \cos(\theta_i)$$

Next, the coordinates of a quantum qubit are calculated and displayed on the Bloch sphere as a white dot. As the polar and azimuthal angles are changed, you will need to rotate the figure to see where the white dot is on the surface of the Bloch sphere.

Note: The reason the angle θ is given as θ/2 is to setup angle between orthogonal basis |0> and |1> so that cos(π/2) = 0, i.e. Prob'ity = 0.

A qubit in superposition can be written as:

$$\Psi(\theta 1, \phi 1) := \cos\left(\frac{\theta 1}{2}\right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \exp(\phi 1) \cdot \sin\left(\frac{\theta 1}{2}\right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Psi(\theta 1, \phi 1) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad YY_{i,j} := \sin(\theta 1) \cdot \sin(\phi 1)$$

$$XX_{i,j} := \sin(\theta 1) \cdot \cos(\phi 1) \qquad ZZ_{i,j} := \cos(\theta 1)$$

We can represent any state by the angles θ, ϕ, γ

$$|\psi\rangle = e^{i\gamma} \left(\cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle \right)$$

|psi> = |0> -> (x, y, z) = (0, 0, 1)

|psi> = |1> -> (x, y, z) = (0, 0, -1)

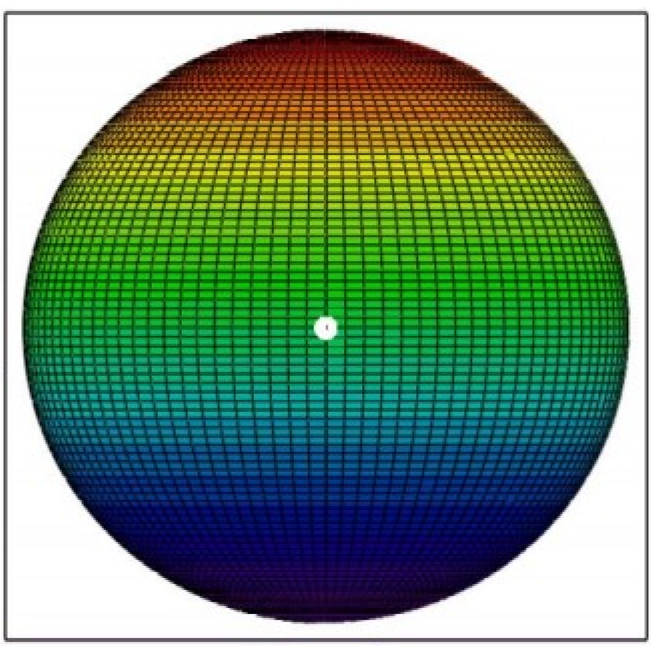
|psi> = (|0> + |1>)/sqrt(2) -> (x, y, z) = (1, 0, 0)

|psi> = (|0> + i|1>)/sqrt(2) -> (x, y, z) = (0, 1, 0

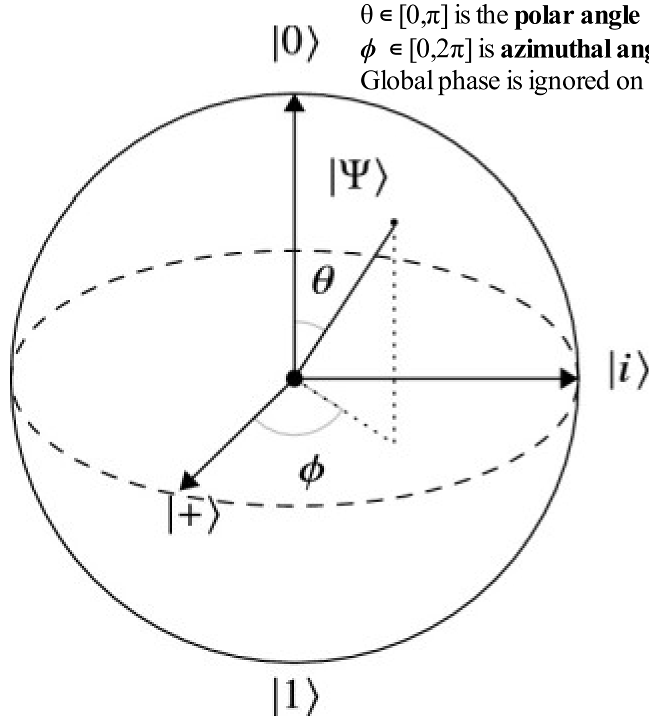
θ ∈ [0,π] is the **polar angle**

ϕ ∈ [0,2π] is **azimuthal angle**

Global phase is ignored on Bloch



(X , Y , Z) , (XX , YY , ZZ)



The Bloch Sphere - Continued

The Bloch sphere is a nice way to visualize quantum states and to <u>identify orthogonal states</u>. Furthermore, because **diametrically opposite states in the Bloch sphere are orthogonal** it also gives insight in which particular states are orthogonal. However, it is **only possible to do this for single qubit states**. When multiple qubit states are considered, t is not possible to visualize states in such a way

Dirac Notation for Bloch Sphere Equatorial +, −, i+, and i–

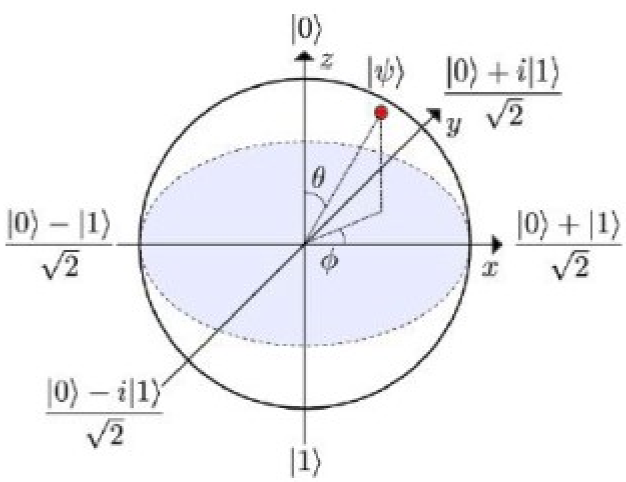
These |+> and |–> states differ by a minus sign on the |1> state. More formally, we call this **difference a relative phase.** The term phase has numerous meanings in physics - in this context, it refers to an angle. The minus sign is related to the angle π (180°) by Euler's identity: $e^{i\pi} = -1$.

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad |i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

Note: This View Flips the "0" |0> |1> Orientation to "1" |1> |0> Orientation

The θ and the ϕ angles are not equivalent in the Bloch sphere. First, they have **different ranges** -- one is π and the other is 2π. More importantly, ϕ is a rotation around a fixed axis, while θ is a rotation around a non-fixed axis that is moving with ϕ. **<u>For ϕ=0 this axis is y</u>**, for **<u>ϕ=π/2 it is x</u>**, and for every other ϕ it is everything in between in the x−y plane.



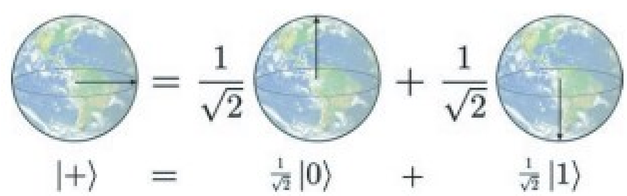
As we can see in above figure, one of the advantages of visualization with the Bloch sphere is that we can **represent superposition of states |0> and |1>**) such as

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

as we see at the X axis. We can also **differentiate** between states that contain **different phases** as is shown in the states along the **X and Y** axes. Let us return to computational universality which we treated above. Now that we have introduced the Bloch sphere, another way to think about a set of gates that satisfies universal computation is one which enables us to reach any point on the Bloch sphere.

See: Quantum Computing: An Applied Approach, Hidary

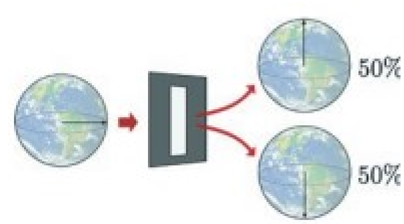
<u>A perfect superposition of |0⟩ and |1⟩</u>
corresponds to the quantum state |+⟩
and points, figuratively speaking,
towards the equator (x-axis) of the sphere.



$$|\uparrow_x\rangle = a|\uparrow_z\rangle + b|\downarrow_z\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

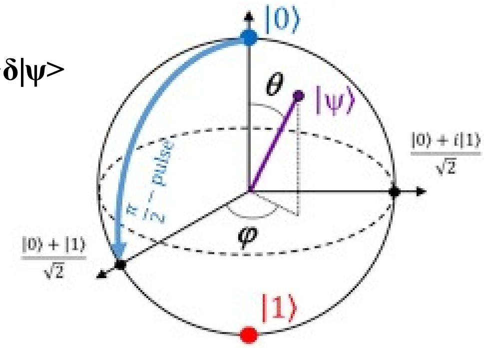
<u>Measuring the "equator" state |+⟩</u>
along the north-south axis results in
"north" or "south" with equal probability.



The Basic Building Blocks of Quantum Computing, Ellerhoff

Expressions for the Bloch Vector Probabilities: Symbol for Bloch Sphere Vector ^n

Let ^n be a Bloch Sphere Basis Vector: $\hat{n} = (\langle\psi|\delta_x|\psi\rangle, \langle\psi|\delta_y|\psi\rangle, \langle\psi|\delta_z|\psi\rangle) = \langle\psi|\hat{\delta}|\psi\rangle$



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Expectation Value: Complex Conjugate

$$\langle\psi|\delta_x|\psi\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\cdot\phi}\cdot\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\cdot\phi}\cdot\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Simplify the Above Expression:

The Square X Matrix Flips the Sign --> $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ----> $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Note: X, θ, ϕ are X_{dot}, θ_{dot}, ϕ_{dot}

Simplify Expression →

$$\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\cdot\phi}\cdot\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\cdot\phi}\cdot\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \rightarrow \cos\left(\frac{\theta}{2}\right)\cdot\sin\left(\frac{\theta}{2}\right)\cdot e^{-\phi\cdot i} + \cos\left(\frac{\theta}{2}\right)\cdot\sin\left(\frac{\theta}{2}\right)\cdot e^{\phi\cdot i}$$

Simplifies To → $\left(e^{\phi\cdot i} + e^{-i\cdot\phi}\right)\cdot\left(\cos\left(\frac{\theta}{2}\right)\cdot\sin\left(\frac{\theta}{2}\right)\right) = 2\cos(\phi)\cdot\left(\cos\left(\frac{\theta}{2}\right)\cdot\sin\left(\frac{\theta}{2}\right)\right) = \sin(\theta)\cdot\cos(\phi)$

Similarly:

$n_x = \sin(\theta)\cdot\cos(\phi)$ $\quad n_y = \sin(\theta)\cdot\sin(\phi)$ $\quad n_z = \cos(\phi)$

$\langle\psi|\delta_x|\psi\rangle = \sin(\theta)\cdot\cos(\phi)$

Thus All States |ψ> can be expressed by some Bloch basis vector ^n, |ψ> = |↑n>

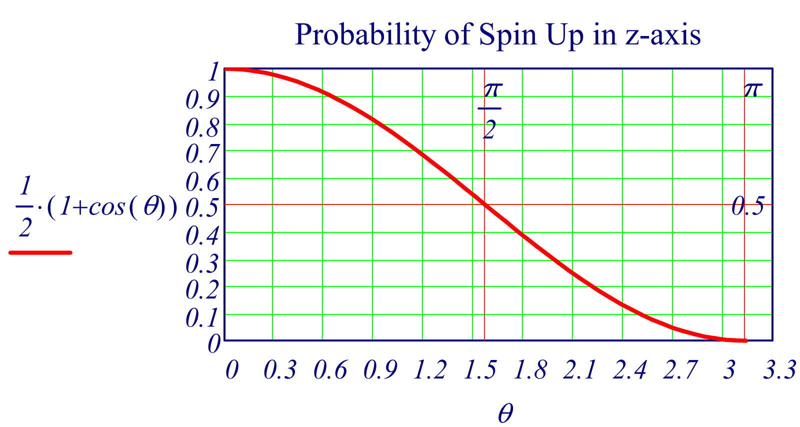
Probability of spin up in z-axis is: $|\langle\uparrow_z|\uparrow_n\rangle|^2 = \cos^2(\theta/2) = 1/2(1+\cos(\theta))$

$|\langle\uparrow_z|\uparrow_n\rangle|^2 = 1/2(1 + \hat{n} * \hat{m})$

1 if ^n*^m = 1

1/2 if ^n*^m = 0

0 if ^n*^m = -1



The Bloch Sphere - Continued

Tutorial: A Course in Quantum Computing, Michael Loceff

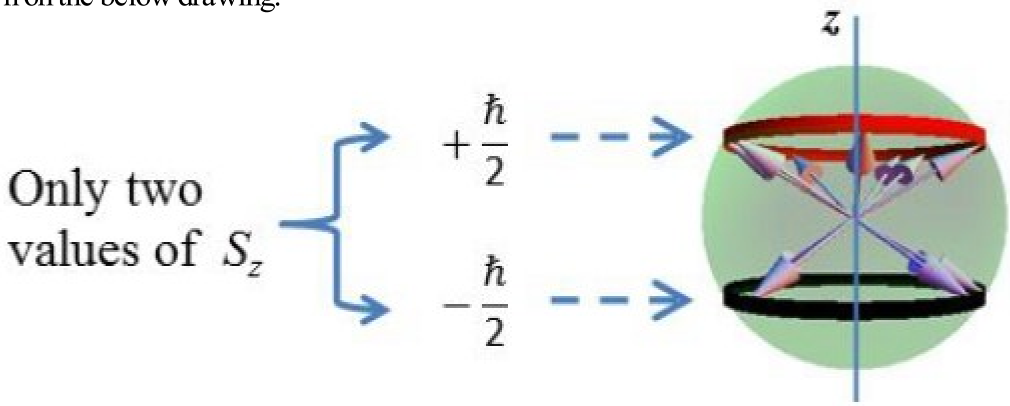
Spin is a property that every electron possesses. Some properties like charge and mass are the same for all electrons, while others like position and momentum vary depending on the electron in question and the exact moment at which we measure. The spin, or more accurate term spin state of an electron has aspects of both. There is an overall magnitude associated with an electron's spin state that does not change. It is represented by the number 1/2, a value shared by all electrons at all times. But then each election can have its own unique vector orientation that varies from electron-to-electron or moment-to-moment.

Spin has a 3-D direction and a scalar magnitude and we can break it into the two aspects, its scalar magnitude,

$$S = |S| = \sqrt{S_x^2 + S_y^2 + S_z^2}$$

and a unit vector that embodies only its orientation (direction) $\frac{S}{|S|}$

S, the spin magnitude, is the same for all electrons under all conditions. Its value is $\frac{\sqrt{3}}{2}\hbar$
where ħ is a tiny number known as Plank's constant.
But, actual measurements of spin along the z-axis only give values of $S_z =$ $\left(+\frac{\hbar}{2}\right)$ or $\left(-\frac{\hbar}{2}\right)$
The situation is shown on the below drawing.



$$0.577 \cdot \sqrt{\frac{3}{2}} = 0.707$$

Why there is electron's projection onto the z-axis is not the entire length of the vector, that is, either straight up at $(+\sqrt{3}/2)\,\hbar$ or straight down at $(-\sqrt{3}/2)\,\hbar$. The electron stubbornly wants to give us only a fraction of that amount, ≈ 57.7%. This corresponds to two groups. The "up group" which forms the angle $\theta \approx 55°$.

This smaller value is due to the Heisenberg Uncertainty Principle. If the spin were to collapse to a state that was any closer to the vertical ±z-axis, we would have too much simultaneous knowledge about its x- and y-components (too close to (0) and its z-component (too close to $(\pm\sqrt{3}/2)\,\hbar$. This would violate the Heisenberg Uncertainty Principle, which requires the combined variation of these observables be larger than a fixed constant. Therefore, S_z must give up some of its claim on the full spin magnitude, $\left(\frac{\sqrt{3}}{2}\right)\hbar$

|+i> and |-i> States. We give a name to the state of the electrons in the (+) group: we call it the $|+\rangle_z$ state (or simply the $|+\rangle$ state, since we consider the z-axis to be the preferred axis in which to project the spin).
We say that the (-) group is in the $|-\rangle_z$ (or just the $|-\rangle$ state. Verbally, these two states are pronounced "plus ket" and "minus ket.

***Embedding of a non-Hermitian Hamiltonian to emulate the von Neumann measurement* scheme**,
Gurpahul Singh, J. Phys. A: Math. Theor. 57 (2024) 035301

The Bloch sphere is a geometrical representation of pure single-qubit states as a point on the unit sphere.
Operations on single qubits commonly used in quantum information processing can be represented on the Bloch sphere. The north pole and the south pole of the Bloch sphere are defined as the orthonormal computational basis states $|0\rangle$ and $|1\rangle$, respectively, and an arbitrary single-qubit pure state, up to the global phase, is represented by a point on the unit sphere, thereby relating the superposition of the basis states to the angular coordinates of the point.
A
The evolution of system state $|\psi(t)\rangle$ as seen on the Bloch sphere. We can see that the initial state ($|+\rangle$ eigenstate) converges to the North pole or $|0\rangle$ state finally. (b) This shows the probability of being in the eigenkets $|0\rangle$ (blue) and $|1\rangle$ (orange) of Hh . The black lines mark the switching on (t_i=3) and switching off (t_f= 7) times for the non-Hermitian part Hnh−(t) . M(0)=m0 , where $m_0 = 1+\epsilon$, ϵ is a very small positive number.

Block Sphere Trajectory with Phase Damping

Parameters

$\theta_0 := \pi / 2$ // Initial polar angle (qubit state)

$\omega := 1$ // Angular frequency

$\gamma := 0.1$ // Phase damping rate

$t := 0, 0.1 .. 20$ // Time range

Damped Bloch vector components

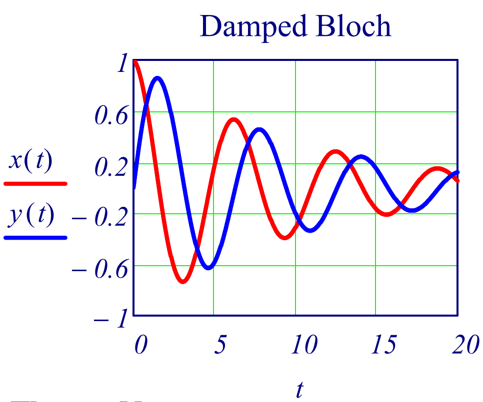
$$\theta_0 := \frac{1}{2} \cdot \pi$$

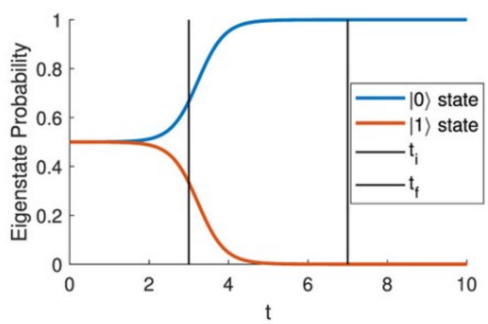
$N := 401 \quad k := 0 .. N \quad \gamma := 0.1 \quad \omega := 1$

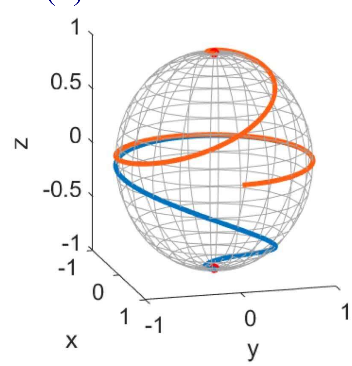
$x(t) := exp(-\gamma \cdot t) \cdot sin(\theta_0) \cdot cos(\omega \cdot t) \qquad X_k := x(k)$

$y(t) := exp(-\gamma \cdot t) \cdot sin(\theta_0) \cdot sin(\omega \cdot t) \qquad Y_k := y(k)$

$(\theta_0) \qquad Z_k := z(0)$







The von-Neumann entropy, is 0 in the beginning, indicating a product state (since $\eta(0) \approx I$). The entanglement increases when the non-Hermiticity is switched on and saturates when $|\psi s(t)\rangle$ has converged to $|0\rangle$.

$$F(t) := e^{-0.2t}$$