VIII C. Explore 4 Qubit Hilbert Space, Operators, and State Evolution

Complex Hilbert Space

$$\mathbb{C}^{2^4} = \mathbb{C}^{16}.$$

Gates, Operators, and States

$$I4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I1 := I$$

$$ket0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad ket1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v0 := ket0 \qquad v1 := ket1$$

Convention for Labelling Row Vector, v[b1,b2,b3,b4]: Binary to Row Indes $Row\_Index = b1\cdot 8 + b2\cdot 4 + b3\cdot 2 + b4$

Let's Make Some Quantum Qubit Information Tools

Create Kronecker, Conjunction, Matrix Exp, Evolution Loop, Apply Qubit to Gate Functions, and GHZ State

Custom Kronecker Function for Math

```
kr(A,B) :=
  $ra \leftarrow rows(A)$
  $ca \leftarrow cols(A)$
  $rb \leftarrow rows(B)$
  $cb \leftarrow cols(B)$
  $f(i,j) \leftarrow 0$
  $C \leftarrow matrix(ra\cdot rb, ca\cdot cb, f)$
  for $i \in 0..rows(A) - 1$
    for $j \in 0..cols(A) - 1$
      for $k \in 0..rows(B) - 1$
        for $l \in 0..cols(B) - 1$
          $row \leftarrow i\cdot rb + k$
          $col \leftarrow j\cdot cb + l$
          $C_{row,col} \leftarrow A_{i,j}\cdot B_{k,l}$
  $C$
```

Complex Conjugate

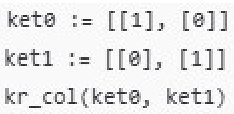
```
conj(M) :=
  $f(i,j) \leftarrow 0$
  $r \leftarrow rows(M)$
  $c \leftarrow cols(M)$
  $C \leftarrow matrix(r,c,f)$
  for $i \in 0..r - 1$
    for $j \in 0..c - 1$
      $C_{i,j} \leftarrow Re(M_{i,j}) - i\cdot Im(M_{i,j})$
  $C$
```

Test Conjugate

$$conj\left(\begin{pmatrix} 1+i & 2-i\cdot 3 \\ 4\cdot i & 5 \end{pmatrix}\right) = \begin{pmatrix} 1-i & 2+3i \\ -4i & 5 \end{pmatrix}$$

$$2 \times 1 \otimes 2 \times 1 = 4 \times 1$$

$$8 \times 1 \otimes 2 \times 1 = 16 \times 1$$



```
ket0 := [[1], [0]]
ket1 := [[0], [1]]
kr_col(ket0, ket1)
```

Apply a single-qubit gate to a 4-qubit system

See Application to Create GHZ4 Below

```
Apply1y1QubitGate4(U,q) :=
  $G \leftarrow kr(U, kr(I, kr(I,I)))$  if $q = 0$
  $G \leftarrow kr(I, kr(U, kr(I,I)))$  if $q = 1$
  $G \leftarrow kr(I, kr(I, kr(U,I)))$  if $q = 2$
  $G \leftarrow kr(I, kr(I, kr(I,U)))$  if $q = 3$
  $G$
```

Each call to kr doubles the dimensionality appropriately—
this creates a 16 × 16 matrix, exactly what is needed for 4-qubit operations.

$$H_{test} := Apply1y1QubitGate4(H,0)$$

// Apply Hadamard to qubit 0

$$Ho := Apply1y1QubitGate4(H,0)$$

Approximate Matrix Exponential

of a Square Matrix Z

$$expM(Z) := \sum_{n=0}^{30} \frac{i\cdot Z^n}{n!}$$

$$expM(Z) = \begin{pmatrix} 2.718i & 0 \\ 0 & 0.368i \end{pmatrix}$$

◆ Function to Calculate Absolute Value of Matrix

$abs(M) :=$

```
$f(i,j) \leftarrow 0$
$A \leftarrow matrix(rows(M), cols(M), f)$
for $i \in 0..rows(M) - 1$
  for $j \in 0..cols(M) - 1$
    $A_{i,j} \leftarrow |M_{i,j}|$
$A$
```

$|i+1| = 1.414$ $abs(H) = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & 0.707 \end{pmatrix}$

◆ Function to Create Identity Matrix for Array N

$Ident(N) :=$

```
$f(i,j) \leftarrow 0$
$I \leftarrow matrix(N, N, f)$
for $i \in 0..rows(I) - 1$
  for $j \in 0..cols(I) - 1$
    $I_{i,j} \leftarrow 1$ if $i = j$
$I$
```

$I_{16} := Ident(16)$ $cols(I_{16}) = 16$

◆ Create Apply CNOT Gate

$ApplyCNOT4(c,t) :=$

```
$G \leftarrow Ident(16)$
for $i \in 0..15$
  $b0 \leftarrow mod\left(floor\left(\frac{i}{8}\right), 2\right)$
  $b1 \leftarrow mod\left(floor\left(\frac{i}{4}\right), 2\right)$
  $b2 \leftarrow mod\left(floor\left(\frac{i}{2}\right), 2\right)$
  $b3 \leftarrow mod(i, 2)$
  $b \leftarrow (b0 \;\; b1 \;\; b2 \;\; b3)^T$
  if $b_c = 1$
    $b_t \leftarrow 1 - b_t$
    $j \leftarrow 8 \cdot b_0 + 4 \cdot b_1 + 2 \cdot b_2 + b_3$
    $G_{i,i} \leftarrow 0$
    $G_{i,j} \leftarrow 1$
$G$
```

$Out := ApplyCNOT4(0,1)$

$rows(Out) = 16$

$cols(Out) = 16$

$CreateSpace(\blacksquare)$

◆ Build GHZ4

$psi0 := kr(kr(kr(v0,v0),v0),v0)$

$H0 := Appy1y1QubitGate4(H,0)$

$psi1 := H0 \cdot psi0$

$CNOT01 := ApplyCNOT4(0,1)$

$CNOT02 := ApplyCNOT4(0,2)$

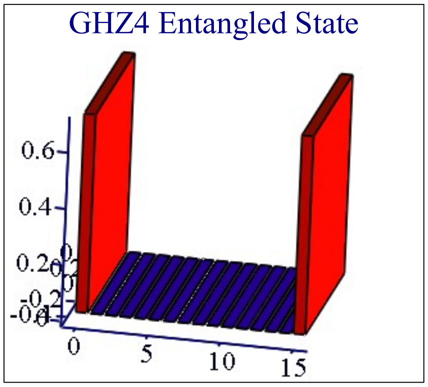
$CNOT03 := ApplyCNOT4(0,3)$

$psi2 := CNOT01 \cdot psi1$

$psi3 := CNOT02 \cdot psi2$

$psi4 := CNOT03 \cdot psi3$

$GHZ4 := psi4$



$GHZ4$

|GHZ4>=(1/sqrt2)(|0000>+|1111>)

GHZ4 State

$psi4^T = (0.707 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0.707)$

Simulation of time evolution of a 4-qubit quantum system starting in a GHZ state

Under the influence of a time-dependent Hamiltonian, dephasing noise, and thermal noise.
The system starts in a GHZ4 state: |GHZ4> = (1/sqrt2)(|0000> + |1111>).
Each qubit state is created using Kronecker products of |0> and |1> vectors

Hamiltonian Evolution:

Time-dependent Hamiltonian:
H(t) = H0 + f(t) * H1
H0 = Z1 + Z2 + Z3 + Z4 (Z-fields on each qubit)
H1 = X1*X2 + X2*X3 + X3*X4 (neighbor coupling)
f(t) = sin(t)

Evolution Operator for Hamiltonian Schoedinger Equation Deterministic Evolution:

U = exp(-i*H(t)*t)

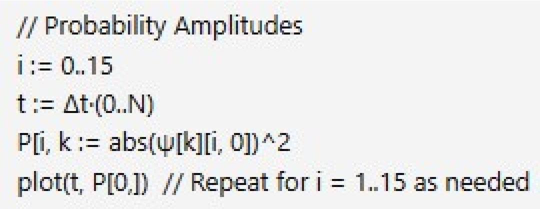
Thermal Mixing:

The normalization factor

$$Z_{\mathrm{th}} = e^{-\beta} + e^{\beta}$$

This is a thermal density matrix for a qubit in thermal equilibrium, assuming energy eigenvalues of -1 and +1, and inverse temperature β=1/kT

Blended with the evolved state:



```
// Probability Amplitudes
i := 0..15
t := Δt·(0..N)
P[i, k := abs(ψ[k][i, 0])^2
plot(t, P[0,]) // Repeat for i = 1..15 as needed
```

Dephasing Noise:

Dephasing on each qubit is modeled with Kraus operators:
K0 = sqrt(1 - p)*I
K1 = sqrt(p)*Z
These are extended to 4-qubits via Kronecker products, then applied to the density matrix: _k -> K_i K_i^

Entropy Calculation:

A partial trace isolates qubit 1:
1 = Tr_{234}()

Then entropy:

S = -1*log2(1) - 2*log2(2)

Probability Tracking:

For each time step k:
P[i, k] = |_k[i]|^2
This gives the probability of observing each basis state |i> over time.

$$p := 0.05 \quad K0 := \sqrt{1-p}\cdot I \quad K1 := \sqrt{p}\cdot Z$$

Kraus operator on each Qubit

$$K0\_1 := kr(K0, kr(I, kr(I, I)))$$
$$K1\_1 := kr(K1, kr(I, kr(I, I)))$$
$$K0\_2 := kr(I, kr(K0, kr(I, I)))$$
$$K1\_2 := kr(I, kr(K1, kr(I, I)))$$
$$K0\_3 := kr(I, kr(I, kr(K0, I)))$$
$$K1\_3 := kr(I, kr(I, kr(K1, I)))$$
$$K0\_4 := kr(I, kr(I, kr(I, K0)))$$
$$K1\_4 := kr(I, kr(I, kr(I, K1)))$$

Qubit Operators

$$X1 := kr(X, kr(I, kr(I, I))) \qquad Z1 := kr(Z, kr(I, kr(I, I)))$$
$$X2 := kr(I, kr(X, kr(I, I))) \qquad Z2 := kr(I, kr(Z, kr(I, I)))$$
$$X3 := kr(I, kr(I, kr(X, I))) \qquad Z3 := kr(I, kr(I, kr(Z, I)))$$
$$X4 := kr(I, kr(I, kr(I, X))) \qquad Z4 := kr(I, kr(I, kr(I, Z)))$$

GHZ4 State

$$GHZ4 := \frac{1}{\sqrt{2}}(kr(ket0, kr(ket0, kr(ket0, ket0))) + kr(ket1, kr(ket1, kr(ket1, ket1))))$$

$$\psi_0 := GHZ4 \qquad U := Ident(16) \qquad T := 0.5 \quad \beta := \frac{1}{T} \qquad Z\_th := \exp(-\beta) + \exp(\beta)$$

Hamiltonian

$$H0 := Z1 + Z2 + Z3 + Z4$$

$$H1 := X1\cdot X2 + X2\cdot X3 + X3\cdot X4$$

$$f(t) := \sin(t)$$

$$H(t) := H0 + f(t)\cdot H1$$

$$\rho\_th := \begin{pmatrix} \frac{\exp(-\beta)}{Z\_th} & 0 \\ 0 & \frac{\exp(\beta)}{Z\_th} \end{pmatrix}$$

$$\rho\_env := kr(\rho\_th, kr(\rho\_th, kr(\rho\_th, \rho\_th)))$$

Quantum Machinery - Refer to Following Page for Implementation

In ideal (closed) quantum systems, evolution is described

by a unitary operator U, such that: $\rho' = U\rho U^\dagger$
But real quantum systems are often open, meaning they interact with their environment.
The evolution becomes non-unitary, and we must use a more general framework.

Kraus Representation Theorem

Any quantum operation (also called a quantum channel) can be written as

$$\rho' = \sum_k K_k \rho K_k^\dagger \qquad \rho' = \sum K_k \rho K_k^\dagger$$

Where: ρ is the input density matrix
Kk are the Kraus operators
They satisfy the completeness relation:

$$\sum_k K_k^\dagger K_k = I$$

This ensures trace is preserved: $\mathrm{Tr}(\rho') = 1$

◆ Physical Interpretation

Each Kk can be thought of as a possible outcome of an interaction with the environment (e.g. a type of noise or measurement effect). The final state is a mixture of these outcomes.

◆ Bit Flip Channel (probability p to flip a qubit)

$$K_0 = \sqrt{1-p} \cdot I, \quad K_1 = \sqrt{p} \cdot X$$

$$\rho' = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger = (1-p)\rho + pX\rho X$$

◆ Phase Damping (dephasing)

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix}$$

◆ Amplitude Damping Channel

Models energy loss (e.g. spontaneous photon emission):

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

◆ Phase Damping (Dephasing)

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix}$$

The Evolution Loop Simulation

The following Program simulates the time evolution of a 4-qubit quantum system starting in a GHZ state, under the influence of a time-dependent Hamiltonian, dephasing noise, and thermal noise. It tracks state evolution (psi_k), density matrices (rho_k), entropy, and probabilities.

Evolution Loop

$\Delta t := 0.01$ $\quad t\_max := 3$ $\quad N := \frac{t\_max}{\Delta t}$

$Hx := kr(Z, kr(I, kr(I, I)))$ $\quad Ux := expM(i \cdot Hx \cdot \Delta t)$

Generate Zero State

Generate |000...0> for n qubits

```
$Zerostate(n) :=$
    $psi \leftarrow v0$
    for $i \in 1..n-1$
        $psi \leftarrow kr(psi, v0)$
    $psi$
```

$\psi_{z4} := Zerostate(4)$

Conventions like Mathcad

Example:

Mathcad defines matrix indices, the state |0000⟩ corresponds to the first basis vector in a 16-element column: $|0000\rangle = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$

|0000⟩ means all four qubits (q_0 to q_3) are in state |0⟩.

It's the ground state (lowest energy, no excitation). In simulations, it's often the starting point for quantum circuits (e.g. GHZ state)

We express this following expression with the kronecker function:

$$|q_0 = 0\rangle \otimes |q_1 = 0\rangle \otimes |q_2 = 0\rangle \otimes |q_3 = 0\rangle$$

$$|0000\rangle = kr(kr(kr(|0\rangle, |0\rangle), |0\rangle), |0\rangle)$$

This I a 16×1 column vector with a 1 in the first position (index 0), representing |0000⟩

$$\psi = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle = |0000\rangle$$

In a 4-qubit system, each qubit has 2 states → total Hilbert space size is: $2^4 = 16$

A full operator acting on all 4 qubits (e.g., a gate matrix, density matrix) must be: 16 x 16

$$\text{abs}(z) = |z| = \sqrt{a^2 + b^2}$$

```
$E :=$
    $k \leftarrow 1$
    $f(i,j) \leftarrow 0$
    $\psi \leftarrow matrix(16, 16, f)$
    $\psi^{\langle 0 \rangle} \leftarrow GHZ4$
    $rho\_all \leftarrow \psi^{\langle 0 \rangle} conj\left(\psi^{\langle 0 \rangle}\right)^T$
    while $k \le 600$
        $Hk \leftarrow H(k \cdot \Delta t)$
        $U \leftarrow expM(i \cdot Hk \cdot \Delta t)$
        $\psi^{\langle k \rangle} \leftarrow U \cdot \psi^{\langle k-1 \rangle}$
        $\rho \leftarrow \psi^{\langle k \rangle} \cdot conj\left(\psi^{\langle k \rangle}\right)^T$
        $A1 \leftarrow K0\_1 \cdot \rho K0\_1^T + K1\_1 \cdot \rho K1\_1^T$
        $A2 \leftarrow K0\_2 \cdot \rho + K1\_2 \cdot \rho K1\_2^T$
        $A3 \leftarrow K0\_3 \cdot \rho K0\_3^T + K1\_3 \cdot \rho K1\_3^T$
        $A4 \leftarrow K0\_4 \cdot \rho K0\_4^T + K1\_4 \cdot \rho K1\_4^T$
        $\rho\_d \leftarrow A1 + A2 + A3 + A4$
        $\alpha \leftarrow exp(-\gamma \cdot \Delta t)$
        $\rho x \leftarrow \alpha \cdot \rho\_d + (1 - \alpha) \cdot \rho\_env$
        $rho\_all \leftarrow stack(rho\_all, \rho)$
        $k \leftarrow k + 1$
    $rho\_all$
```

Evolving the Unitary Transformation, U, under Hamiltonian Hk

Substituted Ux for U

```
$R\_all :=$
    $f(i,j) \leftarrow 0$
    $\psi \leftarrow matrix(16, 16, f)$
    $\psi^{\langle 0 \rangle} \leftarrow GHZ4$
    $rho\_all \leftarrow \psi^{\langle 0 \rangle} conj\left(\psi^{\langle 0 \rangle}\right)^T$
    for $k \in 1..4$
        $\psi^{\langle k \rangle} \leftarrow Ux \cdot \psi^{\langle k-1 \rangle}$
        $rho\_k \leftarrow \psi^{\langle k \rangle} \cdot conj\left(\psi^{\langle k \rangle}\right)^T$
        $rho\_all \leftarrow stack(rho\_all, rho\_k)$
    $rho\_all$
```

$$\psi_{z4}^T = (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)$$

$rows(E) = 9616$ $\quad cols(E) = 16$ $\quad rows(E) \cdot \frac{1}{16} = 601$

Extract kth density matrix: Show 6 of 16 Columns for Iteration k = 600

Selects kth Set of 16 Rows x 16 Columns (Only First 6 Columns are Shown)

$$\rho k(k) := \mathrm{submatrix}(E, 16\cdot k, 16\cdot k - 1 + 9, 0, 0)$$

Note: Ψ States are Complex

$$\rho k(600)^{T} = \begin{pmatrix} 0.441 & 0 & 0 & -0.083 - 0.011i & 0 & 0.018 - 0.136i & 1.052\times 10^{-3} + 0.022i & 0 & 0 \end{pmatrix}$$

```
Prob( Ψall ) :=
    $f(i,j) \leftarrow 0$
    $P \leftarrow \mathrm{matrix}(N+1, 16, f)$
    for $R \in 0..N$
        $\Psi_{in} \leftarrow \mathrm{submatrix}(\Psi_{all}, 16\cdot R, 16\cdot R - 1 + 16, 0, 15)$
        for $k \in 0..15$
            $P_{R,k} \leftarrow \mathrm{Re}\left(\mathrm{abs}\left(\Psi_{in}^{\langle k\rangle}\right)^2\right)$
    $P$
```

$Probability := Prob(E)$ $rows(Probability) = 301$ $cols(Probability) = 16$

Tool to Create Basis Vectors for 4 Qubit States

```
basis_vec_4QB(b3, b2, b1, b0) :=
    $f(i,j) \leftarrow 0$
    $v \leftarrow \mathrm{matrix}(16, 1, f)$
    $k \leftarrow 8\cdot b3 + 4\cdot b2 + 2\cdot b1 + b0$
    $v_k \leftarrow 1$
    $v$
```

Examples

Create Basis Vector for [0,1,0,0]

Create Basis Vector for [1,1,1,1]

$$BV4(b3, b2, b1, b0) := basis\_vec\_4QB(b3, b2, b1, b0)$$

$$BV4(0,1,0,0) = \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} \qquad BV4(1,1,1,1) = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}$$