XII. Gate Identities

***Quantum Computing: An Applied Approach*, Jack Hidary**

$HXH = Z$
$HZH = X$
$HYH = -Y$
$H^\dagger = H = H^{-1}$

$X^2 = Y^2 = Z^2 = I$
$H = (X + Z)/\sqrt{2}$
$H^2 = I$
$SWAP_{12} = C_{12}C_{21}C_{12}$

$$X \cdot Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad Z \cdot X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \cdot Z \cdot H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad -H \cdot Y \cdot H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad H \cdot X \cdot H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

$$H = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix} \qquad H^T = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix} \qquad H^{-1} = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot (X + Z) = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix} \qquad H = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

$$S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad P(\varphi) := \begin{pmatrix} 1 & 0 \\ 0 & e^{\varphi} \end{pmatrix} \qquad R(\varphi) := \begin{pmatrix} 1 & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \qquad T := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \qquad ket0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad SWAP := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad H\_tensor\_I := kronecker(H, I)$$

$C := CNOT$

***Gates, States, and Circuits*, Gavin Crooks**

$$kronecker(X, X) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad kronecker(Y, Y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad kronecker(Z, Z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Bloch Rotation Decomposition

Decompositions of 1-qubit gates into single rotations about a particular axis

$$R_{\vec{n}}(\theta) = e^{-i\frac{1}{2}\theta(n_x X + n_y Y + n_z Z)}$$

$$R_{\vec{n}}(\theta) = \begin{bmatrix} \cos(\frac{1}{2}\theta) - i n_z \sin(\frac{1}{2}\theta) & -n_y \sin(\frac{1}{2}\theta) - i n_x \sin(\frac{1}{2}\theta) \\ n_y \sin(\frac{1}{2}\theta) - i n_x \sin(\frac{1}{2}\theta) & \cos(\frac{1}{2}\theta) + i n_z \sin(\frac{1}{2}\theta) \end{bmatrix}$$

$$R_x(\theta) = R_{\vec{n}}(\theta), \quad \vec{n} = (1, 0, 0)$$
$$R_y(\theta) = R_{\vec{n}}(\theta), \quad \vec{n} = (0, 1, 0)$$
$$R_z(\theta) = R_{\vec{n}}(\theta), \quad \vec{n} = (0, 0, 1)$$

Cofactor, CoF, of a Matrix

Complementary Minor of a Square Matrix

The complementary minor of a matrix A is the matrix $A^{(ij)}$ whose entries $C_{i,j}$ are (n -1) x (n -1) submatrices of A formed by removing the i^{th} row and j^{th} column of A.

Cofactor of a Square Matrix

The cofactor matrix of a matrix A is the matrix C whose entries $C_{i,j}$ are the determinants of the (n -1) x (n -1) submatrices of A formed by removing the i^{th} row and j^{th} column of A. We call $C_{i,j}$ the i, j^{th} cofactor of A, CoF(A, i, j).

Change the Starting Index (ORIGIN) of Arrays from 0 to 1

$ORIGIN := 1$

Function RR: Remove Row R from Matrix M

```
RR(M,R) := | RR ← rows(M)
           | CC ← cols(M)
           | SM ← M_(RR-1, CC)
           | for i ∈ 1 .. RR
           |   for j ∈ 1 .. CC
           |     | SM_(i,j) ← M_(i,j) if i < R
           |     | SM_(i-1,j) ← M_(i,j) if i > R
           | RemR ← SM
```

Function RC: Remove Column C from Matrix M

```
RC(M,C) := | RR ← rows(M)
           | CC ← cols(M)
           | SM ← M_(RR, CC-1)
           | for i ∈ 1 .. RR
           |   for j ∈ 1 .. CC
           |     | SM_(i,j) ← M_(i,j) if j < C
           |     | SM_(i,j-1) ← M_(i,j) if j > C
           | RemC ← SM
```

Verify CoFactor Program, CoF

$$A := \begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 4 \\ 1 & -2 & -5 \end{pmatrix}$$

CoF(A,3,2) = -8

$CoF(A,3,2) = -8$

Calculate CoFactor, CoF, of Matrix M, Row R, Column C

```
CoF(M,R,C) := | SM ← RC(M,C)
              | SM ← RR(SM,R)
              | sign ← (-1)^(R+C)
              | Cof ← sign·|SM|
```

CoFactor Gate Identities

$$SWAP_{12} = C_{12}C_{21}C_{12}$$
$$C_{12}X_1C_{12} = X_1X_2$$
$$C_{12}Y_1C_{12} = Y_1X_2$$
$$C_{12}Z_1C_{12} = Z_1$$
$$C_{12}X_2C_{12} = X_2$$
$$C_{12}Y_2C_{12} = Z_1Y_2$$
$$C_{12}Z_2C_{12} = Z_1Z_2$$
$$R_{z,1}(\theta)C_{12} = C_{12}R_{z,1}(\theta)$$
$$R_{x,2}(\theta)C_{12} = C_{12}R_{x,2}(\theta)$$

$CoF(SWAP,1,2) = 0$

$CoF(C,1,2) \cdot CoF(C,2,1) \cdot CoF(C,1,2) = 0$

$ORIGIN := 0$

Let's Make Some More Quantum Qubit Information Tools

Note: These functions are repeated from section VIII C. The pecularities of Mathcad require that I regenerate them for subsequent use.

Create Kronecker, Conjunction, Matrix Exp, Evolution Loop, Apply Qubit to Gate Functions, and GHZ State

Custom Kronecker Function for Mathcad

$$kr(A,B) := \left|\begin{aligned}
&ra \leftarrow rows(A)\\
&ca \leftarrow cols(A)\\
&rb \leftarrow rows(B)\\
&cb \leftarrow cols(B)\\
&f(i,j) \leftarrow 0\\
&C \leftarrow matrix(ra \cdot rb, ca \cdot cb, f)\\
&for\ \ i \in 0..rows(A)-1\\
&\quad for\ \ j \in 0..cols(A)-1\\
&\quad\quad for\ \ k \in 0..rows(B)-1\\
&\quad\quad\quad for\ \ l \in 0..cols(B)-1\\
&\quad\quad\quad\quad \left|\begin{aligned} &row \leftarrow i \cdot rb + k\\ &col \leftarrow j \cdot cb + l\\ &C_{row,col} \leftarrow A_{i,j} \cdot B_{k,l}\end{aligned}\right.\\
&C
\end{aligned}\right.$$

Complex Conjugate

$$conj(M) := \left|\begin{aligned}
&f(i,j) \leftarrow 0\\
&r \leftarrow rows(M)\\
&c \leftarrow cols(M)\\
&C \leftarrow matrix(r,c,f)\\
&for\ \ i \in 0..r-1\\
&\quad for\ \ j \in 0..c-1\\
&\quad\quad C_{i,j} \leftarrow Re\left(M_{i,j}\right) - \mathrm{i} \cdot Im\left(M_{i,j}\right)\\
&C
\end{aligned}\right.$$

Test Conjugate

$$conj\left(\begin{pmatrix} 1+\mathrm{i} & 2-\mathrm{i}\cdot 3 \\ 4\cdot\mathrm{i} & 5 \end{pmatrix}\right) = \begin{pmatrix} 1-\mathrm{i} & 2+3\mathrm{i} \\ -4\mathrm{i} & 5 \end{pmatrix}$$

Function to Calculate Absolute Value of Matrix

$$abs(M) := \left|\begin{aligned}
&f(i,j) \leftarrow 0\\
&A \leftarrow matrix(rows(M), cols(M), f)\\
&for\ \ i \in 0..rows(M)-1\\
&\quad for\ \ j \in 0..cols(M)-1\\
&\quad\quad A_{i,j} \leftarrow \left|M_{i,j}\right|\\
&A
\end{aligned}\right.$$

Function to Create Identity Matrix for Array N

$$Ident(N) := \left|\begin{aligned}
&f(i,j) \leftarrow 0\\
&I \leftarrow matrix(N,N,f)\\
&for\ \ i \in 0..rows(I)-1\\
&\quad for\ \ j \in 0..cols(I)-1\\
&\quad\quad I_{i,j} \leftarrow 1\ \ if\ \ i = j\\
&I
\end{aligned}\right.$$

$$\psi_{ket}(\theta,\phi) := cos\left(\frac{\theta}{2}\right)\cdot ket0 + exp(\mathrm{i}\cdot\phi)\cdot sin\left(\frac{\theta}{2}\right)\cdot ket1$$

$$|\mathrm{i}+1| = 1.414 \qquad abs(H) = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & 0.707 \end{pmatrix} \qquad I_{16} := Ident(16) \qquad cols\left(I_{16}\right) = 16$$

Exploring - Unary Operators/Functions/Quantum Gates

$Zero := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $One := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

NOT: $X \cdot Zero = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $X \cdot One = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Y Op: $Y \cdot Zero = \begin{pmatrix} 0 \\ i \end{pmatrix}$ $Y \cdot One = \begin{pmatrix} -i \\ 0 \end{pmatrix}$

Z Op: $Z \cdot Zero = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $Z \cdot One = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

S Op: $S \cdot Zero = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $S \cdot One = \begin{pmatrix} 0 \\ i \end{pmatrix}$

Identity: $I \cdot Zero = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $I \cdot One = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

T Op: $T \cdot Zero = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $T \cdot One = \begin{pmatrix} 0 \\ 0.707 + 0.707i \end{pmatrix}$

H Op: $H \cdot Zero = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$ $H \cdot One = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$

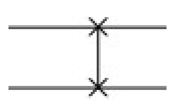
Phase: $R_1(30deg) \cdot Zero = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $R_1(30deg) \cdot One = \begin{pmatrix} 0 \\ 0.592 \end{pmatrix}$

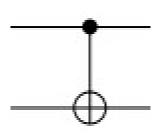
XZ Op: $Z \cdot (X \cdot Zero) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $Z \cdot (X \cdot One) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

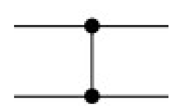
Exploring - Binary Operators/Functions/Quantum Gates/Circuit Models

$OO := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $O1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $SWAP := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $CZ := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Quantum Circuit Diagrams ==>







Explorations of Operators

$'10 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $'11 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $SWAP \cdot O1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $CNOT \cdot '10 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $CZ \cdot '10 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $CZ \cdot '11 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

The **CNOT gate** copies the bit x if y = 0 and gives -x if y = 1, and is the reversible equivalent of the COPY operation. It is reversible because there is a one-to-one correspondence between the initial state & final state. The CNOT operation is a simple permutation of the basis vectors. It can be shown that using single-bit gates $x \rightarrow 1 \oplus x$ or $x \rightarrow \neg$. and the CNOT gate, it is possible to construct only linear functions if we limit ourselves to classical operations.

Ternary Operators: Toffoli and Fredkin Operators - Math

We have discussed both unary and binary operators. Now let's consider the ternary or 3-qubit operators. First, we have the **Toffoli operator**, also known as the **CCNOT gate**. Just as in the CNOT operator, we have control and target qubits. In this case, the first two qubits are control and the third is the target qubit. Both control qubits have to be in state |1> for us to modify the target qubit. Another way of thinking about this is that the first two qubits (x and y) have to satisfy the Boolean AND function — if that equals TRUE then we apply NOT to the target qubit, z. We can represent this action as ,

The nonlinearity of the gate is $(x, y, z) \mapsto (x, y, (z \oplus xy))$ he Toffoli gate performs the NAND operation reversibly. The Toffoil gate can be used to reproduce reversibly all the classical logic circuits. It is a universal gate for all reversible operations of Boolean logic.

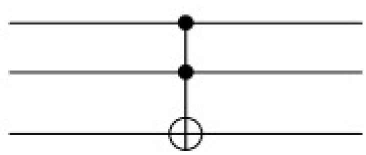
Or as a matrix, CCNOT

$$CCNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

As an example, we apply this gate to the state |110>, 110

$$'110 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad CCNOT \cdot '110 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

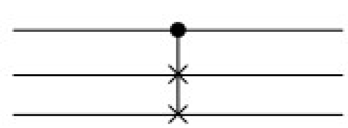
In circuit diagrams, we use the following to denote the Toffoli Gate



Next, let's consider **the Fredkin gate**, also known as the **CSWAP gate**. When we apply this operator, the first qubit is the control and the other two are the target qubits. If the first qubit is in state |0> we do nothing and if it is in state |1> then we SWAP the other two qubits with each other. The matrix representing this operations is

$$CSWAP := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad CSWAP \cdot '110 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

In circuit diagrams we use this symbol for the Fredkin operator



XIII. Visualizing the Difference Between a Superposition and Mixture

Tutorial: *LibreTexts Quantum Tutorials*

The superposition principle, as Feynman said, is at the heart of quantum mechanics. While its mathematical expression is simple, its true meaning is difficult to grasp. For example,

given a linear superposition (not normalized) of two states,

$$|\Psi\rangle = |\phi_1\rangle + |\phi_2\rangle$$

one might assume that it represents a mixture of ϕ_1 and ϕ_2. In other words, half of the quons are in state ϕ_1 and half in ϕ_2. However, the correct quantum mechanical interpretation of this equation is that the system represented by Ψ is **simultaneously in the states ϕ_1 and ϕ_2**, properly weighted.

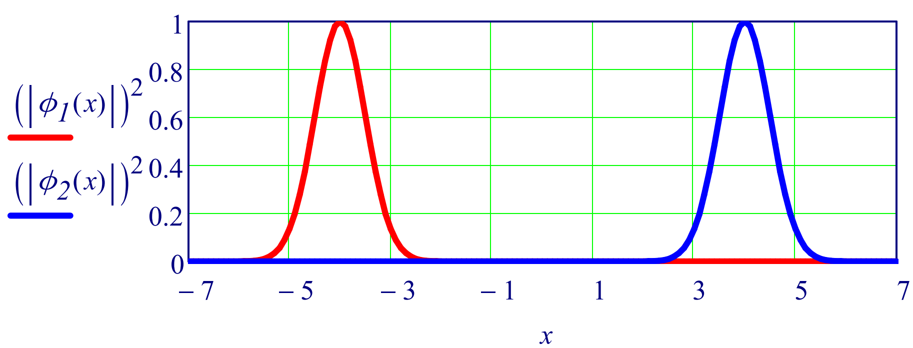
A mixture, half ϕ_1 and half ϕ_2, or any other ratio, **cannot be represented by a wavefunction**. It requires a **density operator**, which is a more general quantum mechanical construct that can be used to represent both pure states (superpositions) and mixtures, as shown below.

$$\hat{\rho}_{\max} = |\Psi\rangle\langle\Psi| \qquad \hat{\rho}_{\min d} = \sum p_i |\Psi_i\rangle\langle\Psi_i|$$

In the equation on the right, p_i is the fraction of the mixture in the state Ψ_i.

To illustrate how these equations distinguish between a mixture and a superposition, we will consider a superposition and a mixture of equally weighted gaussian functions representing one-dimensional wave packets. The normalization constants are omitted in the interest of mathematical clarity. The gaussians are centered at x = ± 4.

$$\phi_1(x) := exp\left[-(x+4)^2\right] \qquad \phi_2(x) := exp\left[-(x-4)^2\right] \qquad x := -7, -6.9 .. 7$$



To visualize how the density operator discriminates between a superposition and a mixture, we calculate its matrix elements in coordinate space for the 5**0-50 superpositio**n and **mixture of ϕ_1 and ϕ_2**.

The superposition is considered first.

$$\Psi(x) := \phi_1(x) + \phi_2(x)$$

The matrix elements of this pure state are calculated as follows.

$$\rho_{\text{pure}} = \langle x|\hat{\rho}_{\text{pure}}|x'\rangle = \langle x|\Psi\rangle\langle\Psi|x'\rangle$$

Looking at the right side we see that the matrix elements are the **product of the probability amplitudes** of a quon in state   being at x and x′. Next we display the density matrix graphically.

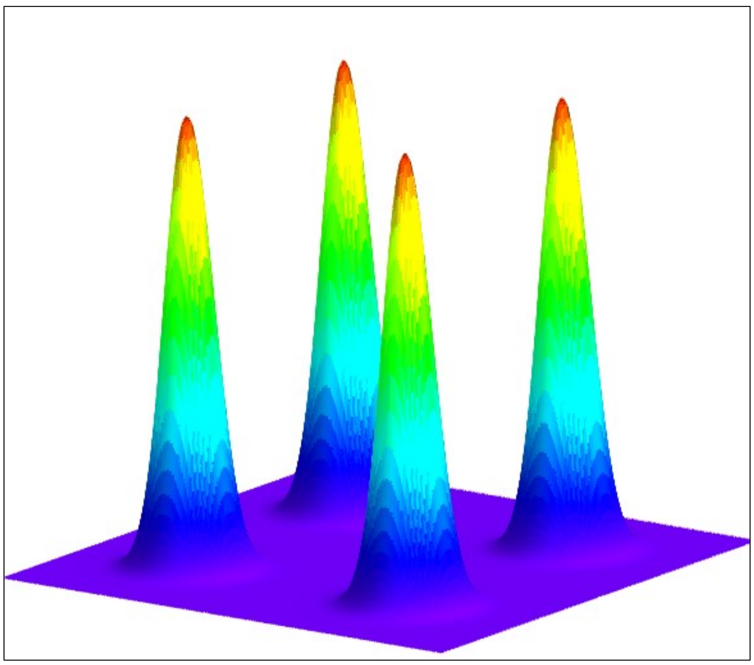
Density Matrix Pure Superposition State

$$DensityMatrixPure(x_1, x_2) := \Psi(x_1)\cdot \Psi(x_2)$$

$x_0 := 8 \quad N := 160 \quad i := 0..N \quad j := 0..N$

$$x1_i := -x_0 + \frac{2\cdot x_0\cdot i}{N} \qquad x2_j := -x_0 + \frac{2\cdot x_0\cdot j}{N}$$

$$DensityMatrixPure_{i,j} := DensityMatrixPure\left(x1_i, x2_j\right)$$



DensityMatrixPure

The presence of off-diagonal elements in this density matrix is the signature of a quantum mechanical superposition. For example,from the quantum mechanical perspective bi-location is possible.

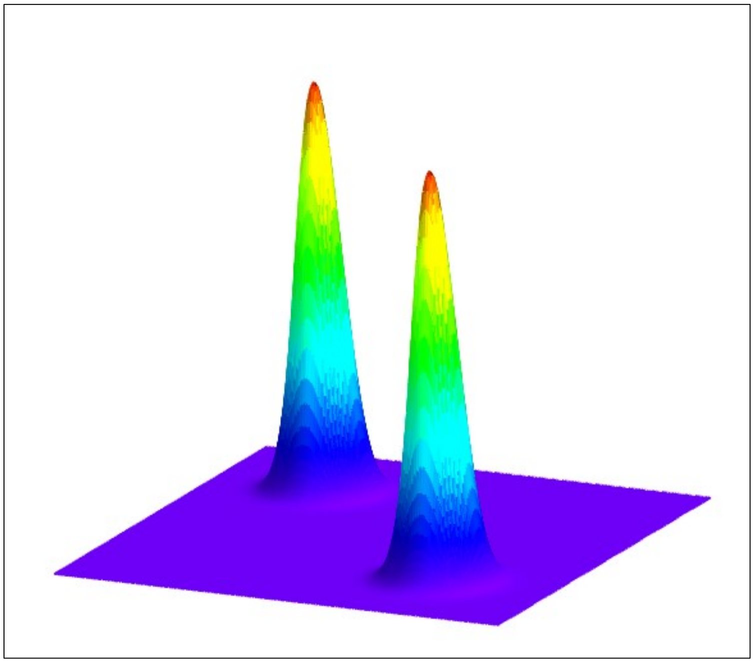
Now we turn our attention to the density matrix of a mixture of gaussian states.

Density Matrix of Mixture State

$$\rho_{\text{mix}} = \langle x\,|\hat{\rho}_{\text{mix}}|\,x'\rangle = \sum_i p_i \langle x|\phi_i\rangle\langle\phi_i|x'\rangle = \frac{1}{2}\langle x|\phi_1\rangle\langle\phi_1|x'\rangle + \frac{1}{2}\langle x|\phi_2\rangle\langle\phi_2|x'\rangle$$

$$DensityMatrixMix(x_1, x_2) := \frac{\phi_1(x_1)\cdot\phi_1(x_2) + \phi_2(x_1)\cdot\phi_2(x_2)}{2}$$

$$DensityMatrixMix_{i,j} := DensityMatrixMix\left(x_{1_i}, x_{2_j}\right)$$



DensityMatrixMix

The obvious difference between a superposition and a mixture is the **absence of off-diagonal elements,**

$\phi_1(x_1)\cdot\phi_1(x_2) + \phi_2(x_1)\cdot\phi_2(x_2)$ in the mixed state.

This indicates the mixture is in a definite but unknown state; it is an example of classical ignorance.

An equivalent way to describe the difference between a superposition and a mixture, is to say that to calculate the probability of measurement outcomes
for a **superposition** you **add the probability amplitudes and square the sum**.
For a **mixture** you **square the individual probability amplitudes** and **sum the squares.**

Nick Herbert (Quantum Reality, page 64) suggested "quon" be used to stand for a generic quantum object. "A quon is any entity, no matter how immense, that exhibits both wave and particle aspects in the peculiar quantum manner.

Visualizing the Difference Between a Superposition and a Mixture - Continued

The Wigner function (**See Section XXIV**) can be used to illustrate the difference between a superposition and a mixture. First consider the following **linear superposition** of Gaussian functions.

$$\Psi(x) := \exp\left[-(x-5)^2\right] + \exp\left[-(x+5)^2\right]$$

The Wigner distribution (See Section XXIV) for this function is calculated and plotted below.

$$W(x,p) := \int_{-\infty}^{\infty} \left[\exp\left[-\left(x+\frac{s}{2}-5\right)^2\right] + \exp\left[-\left(x+\frac{s}{2}+5\right)^2\right]\right] \cdot \exp(i \cdot p \cdot s) \cdot \left[\exp\left[-\left(x-\frac{s}{2}-5\right)^2\right] + \exp\left[-\left(x-\frac{s}{2}+5\right)^2\right]\right] ds$$

Integration yields:

$$W(x,p) := \sqrt{2}\cdot\sqrt{\pi}\cdot\left(2\cdot \exp\left(-2\cdot x^2 - \frac{1}{2}\cdot p^2\right)\cdot \cos(10\cdot p) + \exp\left(-2\cdot x^2 + 20\cdot x - 50 - \frac{1}{2}\cdot p^2\right) \dots + \exp\left(-2\cdot x^2 - 20\cdot x - 50 - \frac{1}{2}\cdot p^2\right)\right)$$

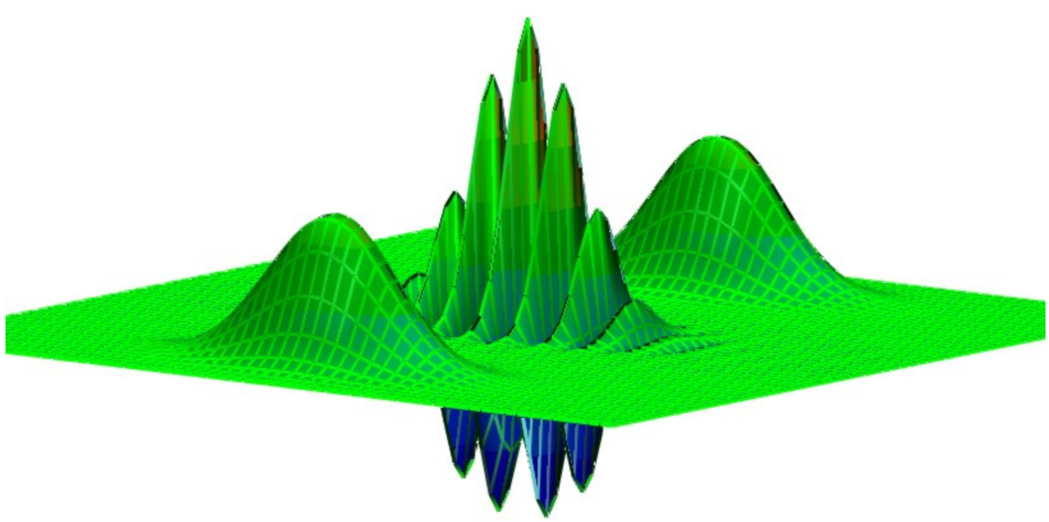
$$N := 60 \qquad i := 0..N \qquad x_i := -7 + \frac{14\cdot i}{N}$$

$$j := 0..N \qquad p_j := -6 + \frac{12\cdot j}{N}$$

$$Wigner_{i,j} := W\left(x_i, p_j\right)$$

The signature of a superposition

the occurrence of interference fringes as seen in the center of the figure below.



Wigner

XIV. Two-Photon Interference

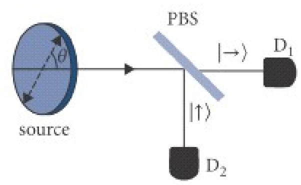
Polarization Beam Splitter and Pockel Cell

Tutorial: *Quantum Mechanics for Beginners,* M. Suhail Zubairy, Chapter 9.5

In this section, we discuss how we can measure the polarization state. A polarizer is an inconvenient device as the photon is either transmitted or it is absorbed. What is more desirable is a device that is able to send one polarization state (say $|\rightarrow\rangle$) along one way and the other $|\uparrow\rangle$ along a different path. This is done in a polarization beam splitter.

Let us consider a photon that is prepared in the polarization state

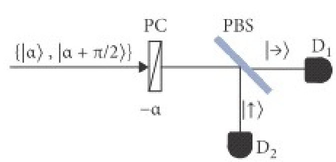
$$|\theta\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle.$$



If a photon polarized in a direction making an angle θ with the polarization axis is incident on a polarization beam splitter (PBS), it can pass through as a photon in state $|\rightarrow\rangle$ in the forward direction or get reflected in state $|\uparrow\rangle$ in the downward direction.

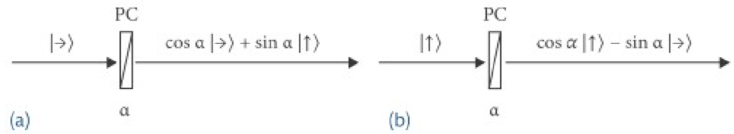
When such a photon is incident on a polarization beam splitter, as shown in the above figure, it can either go in the forward direction in the horizontally polarized state $|\rightarrow\rangle$ or in the downward direction in the vertically polarized state $|\uparrow\rangle$. We can then find the state of the photon depending on whether we get a click at the detector D1 or at the detector D2 . A click at D1 means that the photon is in the state $|\rightarrow\rangle$ and a click at D2 means that the photon is in the state $|\uparrow\rangle$. The probability of the click at D1 is $|\langle\rightarrow|\theta\rangle|^2 = \cos^2\theta$ and the probability of click at D2 is $|\langle\uparrow|\theta\rangle|^2 = \sin^2\theta$.

Unlike a polarizer, the polarizing beam splitter cannot be easily rotated to measure the polarization state along some other axis, say along an axis rotated by an angle α with the horizontal. For that, we need to insert a polarization rotator before the polarizing beam splitter. This device should be able to rotate the state of the polarization of an incoming photon by an angle α before passing it through the polarization beam splitter.



If a photon polarized in a direction making an angle θ with the polarization axis is incident on a polarization beam splitter (PBS), it can pass through as a photon in state $|\rightarrow\rangle$ in the forward direction or get reflected in state $|\uparrow\rangle$ in the downward direction.

One such device is the Pockel cell. It is an electro-optic device which rotates the polarization of the incident light passing



A Pockel cell that rotates the polarization by angle $-\alpha$ followed by a beam splitter can determine the polarization of the incoming photon. A click at D1 implies that the polarization of the incoming photon is along an angle α with the horizontal and a click at D2 means that the state of the incoming photon is along an angle $\alpha + \pi/2$ with the horizontal.

through it in proportion to the applied voltage. As an example, by applying an appropriate voltage, the Pockel cell can rotate the polarization of a photon by an angle α with the horizontal as shown in . As a result the horizontally and vertically polarized photons in states $|\rightarrow\rangle$ and $|\uparrow\rangle$, respectively, undergo the following transformations:

$$|\rightarrow\rangle \rightarrow |+\alpha\rangle \equiv |\,\alpha\,\rangle = \cos\alpha\,|\rightarrow\rangle + \sin\alpha\,|\uparrow\rangle$$
$$|\uparrow\rangle \;\rightarrow |-\alpha\rangle \equiv |\,\alpha + \pi/2\,\rangle = \cos\alpha\,|\uparrow\rangle - \sin\alpha\,|\rightarrow\rangle.$$

We note that, just like the pair of states $\{|\rightarrow\rangle, |\uparrow\rangle\}$, the states $\{|+\alpha\rangle, |-\alpha\rangle\}$ are normalized and are mutually orthogonal, i.e.,

$$\langle +\alpha\,|+\alpha\,\rangle = \langle -\alpha\,|-\alpha\,\rangle = 1,$$
$$\langle +\alpha\,|-\alpha\,\rangle = \langle -\alpha\,|+\alpha\,\rangle = 0.$$

A polarization beam splitter can determine whether the polarization state of the incoming photon is $|\rightarrow\rangle$ or $|\uparrow\rangle$. A question of interest is: How can we determine whether the polarization of the incoming photon is along an angle α or along $\alpha + \pi/2$ with the horizontal? The corresponding states are $|+\alpha\,\rangle \equiv |\alpha\rangle$ and $|-\alpha\,\rangle \equiv |\,\alpha + \pi/2\rangle$. A way of doing this is to first rotate the polarization angle of the incoming photon by an angle $-\alpha$. This should transform the state $|\alpha\rangle$ to $|\rightarrow\rangle$ and the state $|\alpha + \pi/2\rangle$ to $|\uparrow\rangle$. This can be done by passing the photon through a Pockel cell that rotates the polarization by an angle $-\alpha$ with the horizontal. Next the photon passes through a polarization beam splitter as shown in the Figure below. If the detector D1 clicks, the polarization of the incoming photon is along an angle α with the horizontal (in state $|\alpha\rangle$) and a click at the detector D 2 means that the state of the incoming photon is along an angle $\alpha + \pi/2$ with the horizontal (in state $|\alpha + \pi/2\rangle$).

As an example, if we want to find whether the photon is in the state $|\,\theta = 45^\circ\,\rangle \equiv |\nearrow\rangle$ or $|\,\theta = 135^\circ\,\rangle \equiv |\nwarrow\rangle$, we consider the set-up in Fig. 9.14 with $\alpha = 45^\circ$. A rotation of the polarization by an angle -45° transforms the state $|\theta = 45^\circ\,\rangle \equiv |\nearrow\rangle$ to the horizontally polarized state $|\rightarrow\rangle$ and the state $|\theta = 135^\circ\,\rangle \equiv |\nwarrow\rangle$ to the vertically polarized state $|\uparrow\rangle$. Therefore a click at D1 implies that the incoming photon is in the state $|\nearrow\rangle$ and a click at D2 implies that the incoming photon is in the state $|\nwarrow\rangle$.