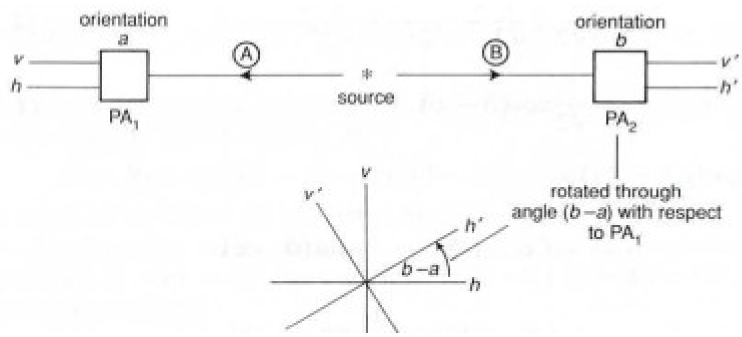
XV. A Proof of Bell's Theorem and Bell State Preparation

This Analysis is Based Jim Baggott's analysis of Bell's theorem as presented in Chapter 4 of ***The Meaning of Quantum Theory*** & Dr. Frank Rioux's Methodology using matrix and tensor algebra.

A two-stage atomic cascade emits entangled photons (A and B) in opposite directions with the same circular polarization according to observers in their path.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A |L\rangle_B + |R\rangle_A |R\rangle_B\right]$$

The experiment involves the measurement of photon polarization states in the vertical/horizontal measurement basis, and allows for the rotation of the right-hand detector through an angle of θ, in order to explore the consequences of quantum mechanical entanglement. PA stands for polarization analyzer and could simply be a calcite crystal.



In vector notation the left- and right-circular polarization states are expressed as follows:

Left circular polarization: $L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$ Right circular polarization: $R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Tensor Product of Column Vectors, In tensor notation the initial state is the following entangled superposition:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A |L\rangle_B + |R\rangle_A |R\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ i \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}_B + \begin{pmatrix} 1 \\ -i \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

However, as mentioned above, the photon polarization measurements will actually be made in the vertical/horizontal basis. These polarization measurement states for photons A and B in vector representation are given below. Θ is the angle through which the PA_2 has been rotated.

Vertical Polarization: $V_A := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_B := \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ Horizontal Polarization: $H_A := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $H_B := \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$

It is easy to show that |Ψ> in the vertical/horizontal basis is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_A |V\rangle_B - |H\rangle_A |H\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

There are four possible measurement outcomes: both photons are vertically polarized, both are horizontally polarized, one is vertical and the other horizontal, and vice versa. The vector representations of the measurement states are obtained by tensor multiplication of the individual photon states.

$$|V_A V_B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \\ 0 \end{pmatrix} \qquad |V_A H_B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \\ 0 \end{pmatrix}$$

$$|H_A V_B\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix} \qquad |H_A H_B\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$$

The initial state and the measurement eigenstates are written in Mathcad syntax.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad VaVb(\theta) := \begin{pmatrix} cos(\theta) \\ -sin(\theta) \\ 0 \\ 0 \end{pmatrix} \qquad VaHb(\theta) := \begin{pmatrix} sin(\theta) \\ cos(\theta) \\ 0 \\ 0 \end{pmatrix}$$

$$HaVb(\theta) := \begin{pmatrix} 0 \\ 0 \\ cos(\theta) \\ -sin(\theta) \end{pmatrix} \qquad HaHb(\theta) := \begin{pmatrix} 0 \\ 0 \\ sin(\theta) \\ cos(\theta) \end{pmatrix}$$

The projections of the initial state onto the four measurement states are,

Note: θ.dot --> $\theta := \pi$

Probability Amplitude:

$$\begin{pmatrix} VaVb(\theta)^T \cdot \Psi \\ VaHb(\theta)^T \cdot \Psi \\ HaVb(\theta)^T \cdot \Psi \\ HaHb(\theta)^T \cdot \Psi \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Probability:

$$\begin{bmatrix} \left(VaVb(\theta)^T \cdot \Psi\right)^2 \\ \left(VaHb(\theta)^T \cdot \Psi\right)^2 \\ \left(HaVb(\theta)^T \cdot \Psi\right)^2 \\ \left(HaHb(\theta)^T \cdot \Psi\right)^2 \end{bmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Assigning an eigenvalue of +1 to a vertical polarization measurement and -1 to a horizontal polarization measurement allows the calculation of the expectation value for the joint polarization measurements, a function which quantifies the correlation between the joint measurements. The eigenvalues for the four joint measurement outcomes are: VaVb = 1; VaHb = -1; HaVb = -1; HaHb = 1. Weighting these by the probability of their occurrence gives the **Expectation Value, E(θ)** or **Correlation Function.**

$$E(\theta) := \left(VaVb(\theta)^T \cdot \Psi\right)^2 - \left(VaHb(\theta)^T \cdot \Psi\right)^2 - \left(HaVb(\theta)^T \cdot \Psi\right)^2 + \left(HaHb(\theta)^T \cdot \Psi\right)^2 \rightarrow \cos(\theta)^2 - \sin(\theta)^2$$

$$\rightarrow \cos(2\cdot\theta)$$

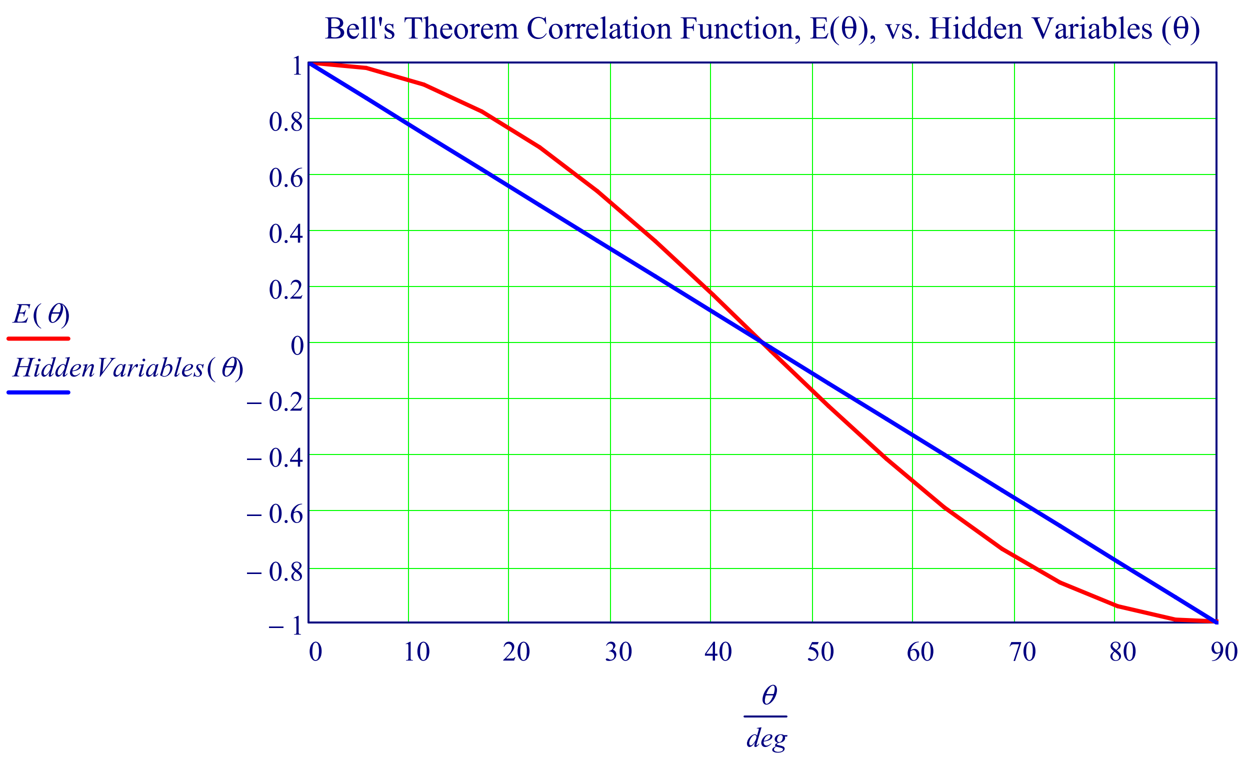
As shown above the evaluation of E(θ) yields cos(2θ). For $\theta = 0^0$ there is perfect correlation; for $\theta = 90^0$ perfect anti-correlation; for $\theta = 45^0$ no correlation.

$$E(0\cdot deg) = 1 \qquad E(90\cdot deg) = -1 \qquad E(45\cdot deg) = 0$$

Baggott presented a correlation function for this experiment based on a local hidden variable model of reality (pp. 110-113, 127-131). It (linear blue line) and the quantum mechanical correlation function, E(θ), are compared on the graph below. Quantum theory and local realism disagree at all angles except 0, 45, and 90 degrees.

$$HiddenVariables(\theta) := 1 - \frac{\theta}{45deg}$$

<u>Comparison of Quantum Theory Expectation vs. Local Hidden Variables</u>



Mathcad Bell States and Bell's CHSH Theorem

There are four Bell states, each representing a **maximally entangled two-qubit state**.
You can generate each by modifying the input to the Bell circuit (either by flipping qubits before or after the Hadamard + CNOT gates). Here's how to generate all four in Mathcad, **using the kronecker() function**.

How to Prepare Bell States

$H\_tensor\_I := kronecker(H, I)$

This represents the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Each is a 4×1 column vector in the computational basis:

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Generate Each Bell State

Plus

a. $\langle\phi+\rangle = (|0\oplus0\rangle + |1\oplus1\rangle) /\sqrt{2}$

$\psi_0 := (1 \ \ 0 \ \ 0 \ \ 0)^T$ $\quad \psi_1 := H\_tensor\_I \cdot \psi_0$

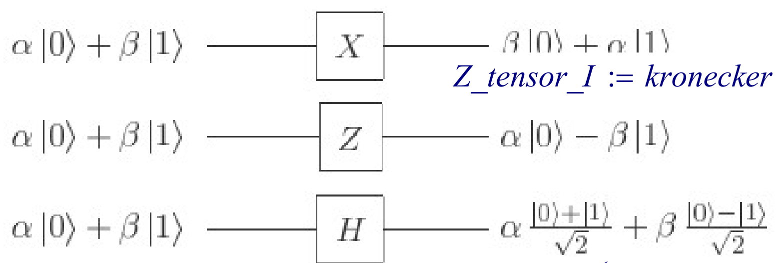
$\psi\_\phi\_plus := CNOT \cdot \psi_1$ $\quad \psi_1^T = (0.707 \ \ 0 \ \ 0.707 \ \ 0)$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Minus

b. $\langle\phi^-\rangle = (|0\oplus0\rangle - |1\oplus1\rangle) /\sqrt{2}$

Flip second qubit after entanglement



$Z\_tensor\_I := kronecker(I, Z)$

$\psi\_\phi\_minus := Z\_tensor\_I \cdot \psi\_\phi\_plus$

$\psi\_\phi\_minus^T = (0.707 \ \ 0 \ \ 0 \ \ -0.707)$

$$Z\_tensor\_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Opposite Plus

c. $\langle\psi^+\rangle = (|0\oplus1\rangle + |1\oplus0\rangle) /\sqrt{2}$

Start with $|01\rangle = |0\rangle \otimes |1\rangle$

$ket1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad ket0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \psi 0 := kronecker(diag(ket0), diag(ket1))$

$$\psi 0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\psi 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \end{pmatrix}$$

$$\psi\_\psi\_plus := CNOT \cdot \psi 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\psi := \frac{1}{\sqrt{2}}$$

Opposite Minus

d. $\langle\psi^-\rangle = (|0\oplus1\rangle - |1\oplus0\rangle) /\sqrt{2}$

$Z\_tensor\_I := kronecker(Z, I)$

$$Z\_tensor\_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\psi\_\psi\_minus := Z\_tensor\_I \cdot \psi\_\psi\_plus$

$$\psi\_\psi\_minus = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 \\ 0 & -0.707 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Each psi_* vector will output the corresponding Bell state

Summary of Bell's Theorem using the CHSH Approach

With hidden variable theory, we consider the correlation of spin measurements of EPR pair at different positions. A and B make measurements with detectors along directions of two vectors ***a*** and ***b***, respectively. Let A and B be results of the measurements. These results depend on not only the direction but also the hidden variable ψ as A(a, ψ) and B(b, ψ). When directions of spin operator s are measured, the values A and B take the maximum to be 1 and the minimum to be -1. Thus the conditions

CHSH: Clauser, Horne, Shimony, and Holt

1. Define Pauli matrices $\sigma_x := X \quad \sigma_y := Y \quad \sigma_z := Z$

$$\psi := \frac{1}{\sqrt{2}}\cdot(TP(Up, Up) + TP(Dn, Dn))$$

2. Define the Bell state (|Φ⁺⟩)

$$\sigma_x^{\,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \psi := \frac{1}{\sqrt{2}}\cdot Z\_tensor\_I\cdot.$$

3. Define measurement directions

We'll use observables as spin projections along directions a, a_θ, b, b_θ.

Choose 4 measurement angles: $\theta a := 0 \quad \theta a_\theta := \frac{\pi}{2} \quad \theta b := \frac{\pi}{4} \quad \theta b_\theta := \frac{-\pi}{4}$

Now define spin observables along angle θ: $\sigma(\theta) := cos(\theta)\cdot Z + sin(\theta)\cdot X$

$$\psi := \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix}$$

4. Define tensor products for each pair of observables

$A := \sigma(\theta a) \quad A_\theta := \sigma(\theta a_\theta) \quad B := \sigma(\theta b) \quad B_\theta := \sigma(\theta b_\theta)$

$A\_tensor\_B := kronecker(A, B) \qquad A\_tensor\_B_\theta := kronecker(A, B_\theta)$

$A\theta\_tensor\_B := kronecker(A_\theta, B) \qquad A\theta\_tensor\_B\theta := kronecker(A_\theta, B_\theta)$

5. Expectation values

Each expectation value is:

$Eab := \psi^T\cdot A\_tensor\_B\cdot\psi \qquad Ea\theta b := \psi^T\cdot A\theta\_tensor\_B\cdot\psi$

$Eab_\theta := \psi^T\cdot A\_tensor\_B_\theta\cdot\psi \qquad Ea\theta b_\theta := \psi^T\cdot A\theta\_tensor\_B\theta\cdot\psi$

6. Compute the CHSH expression

$S := Eab + Eab_\theta + Ea\theta b - Ea\theta b_\theta \qquad S = 2.828$

Bell's Inequality: $\mathbf{P(\uparrow_a, \uparrow_b) <= P(\uparrow_a, \uparrow_c) + P(\uparrow_c, \uparrow_b)}$

We consider now a bipartite system: $\mathcal{H} = \mathcal{H}A \otimes \mathcal{H}B$. The total angular momentum is

$$L_T = L_A + L_B, \qquad [L_A, L_B] = 0 \qquad L_A := 1 \qquad L_B := 1$$

As unitary operators, we choose

$$\mathcal{A} = e^{i\alpha L_A}, \qquad \mathcal{A}' = e^{i\alpha' L_A}, \qquad \mathcal{B} = e^{i\beta L_B}, \qquad \mathcal{B}' = e^{i\beta L_B},$$

where $(a, a_\theta, \beta, \beta_\theta)$ are arbitrary real parameters which play a role akin to that of the four Bell's angles. Denoting by $|m_A, m_B> = |m_A>|m_B>$ the eigenvectors of L_T, as entangled state $|\psi>$ we shall take the state with vanishing total angular momentum given by

Use the angles: $\alpha := 0 \quad \alpha_\theta := \frac{\pi}{2} \quad \beta := \frac{\pi}{4} \quad \beta_\theta := \frac{3\pi}{4}$

$$C_L := \left(exp(i\cdot\alpha\cdot L_A) + exp(i\cdot\alpha_\theta L_A)\right)\cdot exp(i\cdot\beta\cdot L_B) + \left(exp(i\cdot\alpha\cdot L_A) - exp(i\cdot\alpha_\theta L_A)\right)\cdot exp(i\cdot\beta_\theta\cdot L_B) = 2.828i$$

Another derivation for spin correlation, cs: $c_1^{\,2} + c_2^{\,2} = 1$

Thus the Bell inequality breaks down in quantum mechanics.

1. Diagonal Matrix from a Column Vector

$|v\rangle\langle v|.$

If you want to place the vector's elements along the diagonal, use diag() in Mathcad:

v := [1, 0, 0, 0] // a 1×4 row vector, or transpose a column vector
D := diag(v)

This is equivalent to:

◆ Interpretation: `ket0` = $|00\rangle$

ket00 := [[1], [0], [0], [0]] // 4×1 column vector

$$ket00 := \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Solution: Embedding Vectors in Square Matrices

To use kronecker() with Mathcad's limitations, here's how you can embed vectors as square matrices, apply the Kronecker product, and then extract the desired result:

◇ Step 1: Define state vectors as diagonal matrices

This makes them 2×2 square matrices so kronecker() can accept them

ket0 := diag([1, 0]) // Becomes 2×2: [[1, 0], [0, 0]]
ket1 := diag([0, 1])

$$ket0 := diag\left(\begin{pmatrix}1\\0\end{pmatrix}\right) \qquad ket0 = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$ket0 = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \qquad ket0 = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$diag(ket0d) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$kronecker(diag(ket0d), diag(ket1d)) = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

◈ ◍ What if you want state vectors?

If you're trying to get 4×1 state vectors, you'll need to bypass kronecker() and build them manually:

ket00 := [[1], [0], [0], [0]] // |00⟩
ket01 := [[0], [1], [0], [0]] // |01⟩
ket10 := [[0], [0], [1], [0]] // |10⟩
ket11 := [[0], [0], [0], [1]] // |11⟩

$$ket00 := \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad ket01 := \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad ket10 := \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad ket11 := \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$dg(x) := diag(x)$$

$$dg(ket00) = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \quad dg(ket01) = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \quad dg(ket10) = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix} \quad dg(ket11) = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

Bell States and Their 4×4 Projection Matrices

Let's define the four Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Each is a 4×1 column vector in the computational basis: $|00\rangle, |01\rangle, |10\rangle, |11$

Step 1: Define an arbitrary 2-qubit state

```
psi := [ [1], [0], [0], [0] ]  // |00⟩ state
```
This is a 4×1 column vector.

Step 2: Define Bell States as Vectors

```
ket00 := [ [1], [0], [0], [0] ]
ket01 := [ [0], [1], [0], [0] ]
ket10 := [ [0], [0], [1], [0] ]
ket11 := [ [0], [0], [0], [1] ]

Φplus  :=  (1/√2) * (ket00 + ket11)
Φminus :=  (1/√2) * (ket00 - ket11)
Ψplus  :=  (1/√2) * (ket01 + ket10)
Ψminus :=  (1/√2) * (ket01 - ket10)
```

Step 3: Compute 4×4 Projection Operators

$$P = |\mathrm{Bell}\rangle\langle\mathrm{Bell}|$$

```
PΦplus  := Φplus * transpose(Φplus)
PΦminus := Φminus * transpose(Φminus)
PΨplus  := Ψplus * transpose(Ψplus)
PΨminus := Ψminus * transpose(Ψminus)
```

Step 4: Compute probabilities of measuring each Bell state

```
prob_Φplus  := transpose(psi) * PΦplus  * psi
prob_Φminus := transpose(psi) * PΦminus * psi
prob_Ψplus  := transpose(psi) * PΨplus  * psi
prob_Ψminus := transpose(psi) * PΨminus * psi
```